There are many practically important optimization problems that are NP-hard.

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- Exploit special structure of instances occurring in practise.
- Consider algorithms that do not compute the optimal solution but provide solutions that are close to optimum.

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Why approximation algorithms?

- We need algorithms for hard problems.
- It gives a rigorous mathematical base for studying heuristics.
- It provides a metric to compare the difficulty of various optimization problems.
- Proving theorems may give a deeper theoretical understanding which in turn leads to new algorithmic approaches.

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An optimization problem $P = (\mathcal{I}, \mathrm{sol}, m, \mathrm{goal})$ is in **NPO** if

- $x \in \mathcal{I}$ can be decided in polynomial time
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$$R(x,y) := \max\left\{\frac{m(x,y)}{m^*(x)}, \frac{m^*(x)}{m(x,y)}\right\}$$

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An algorithm A is an r-approximation algorithm iff

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A PTAS for a problem P from NPO is an algorithm that takes as input $x \in \mathcal{I}$ and $\epsilon > 0$ and produces a solution y for x with

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- Set Cover
- Minimum Multicut
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Class APX not important in practise.

Instead of saying problem P is in APX one says problem P admits a 4-approximation.

One only says that a problem is APX-hard.

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