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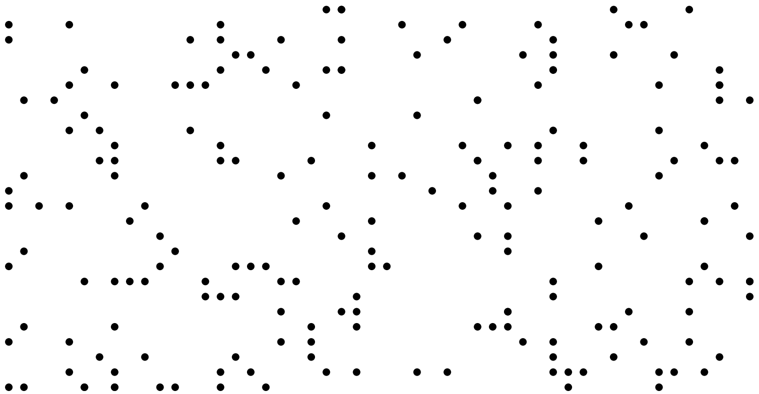

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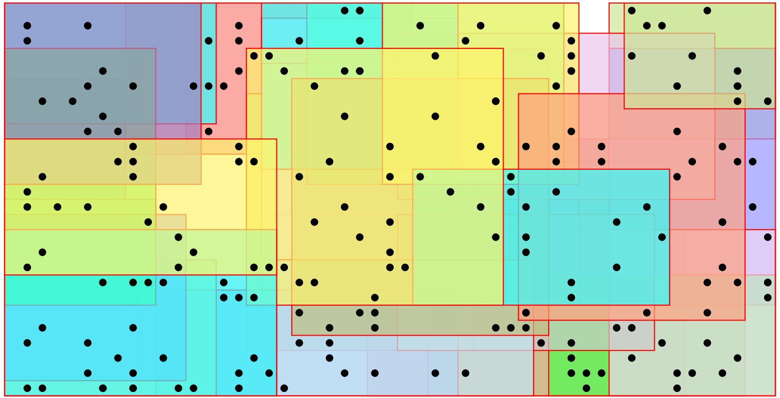

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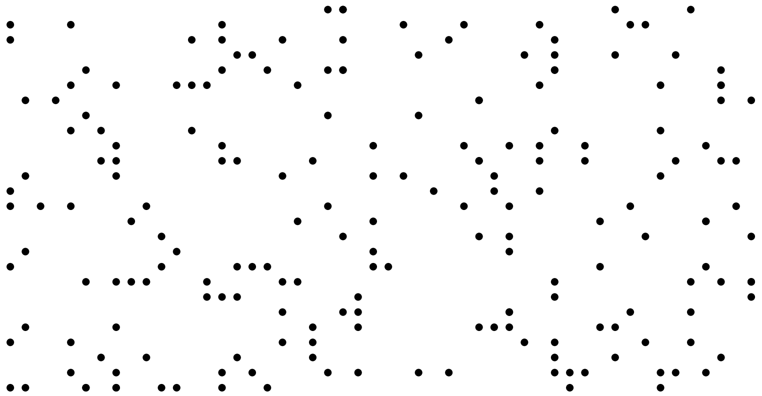

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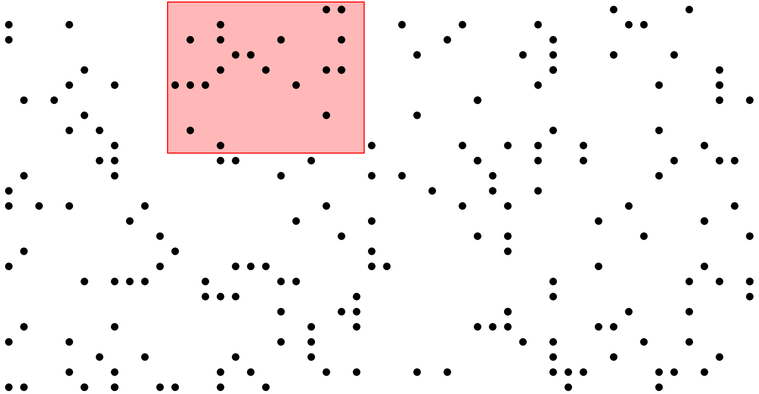

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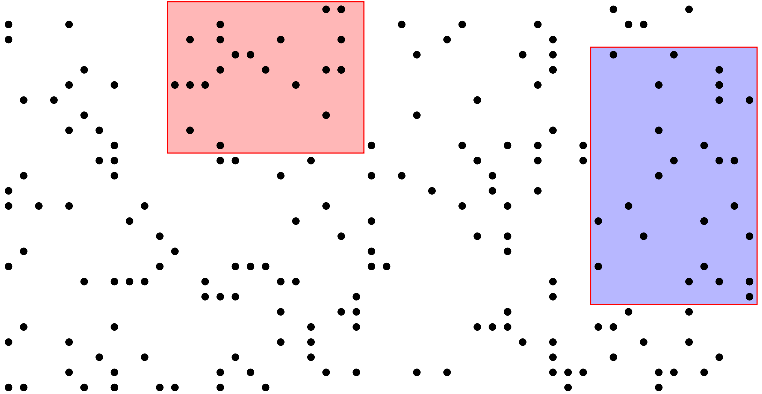

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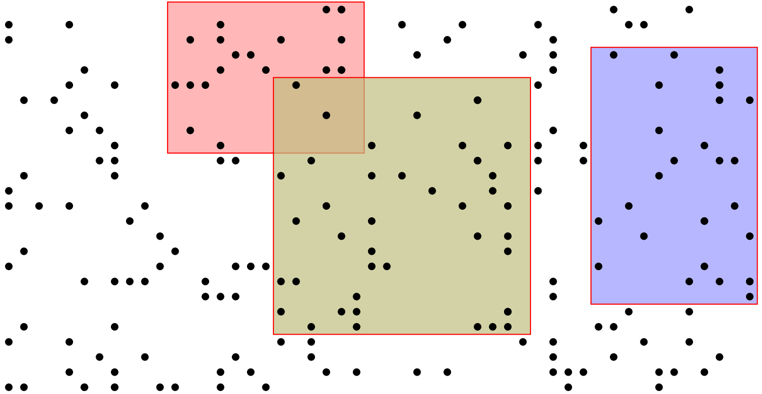

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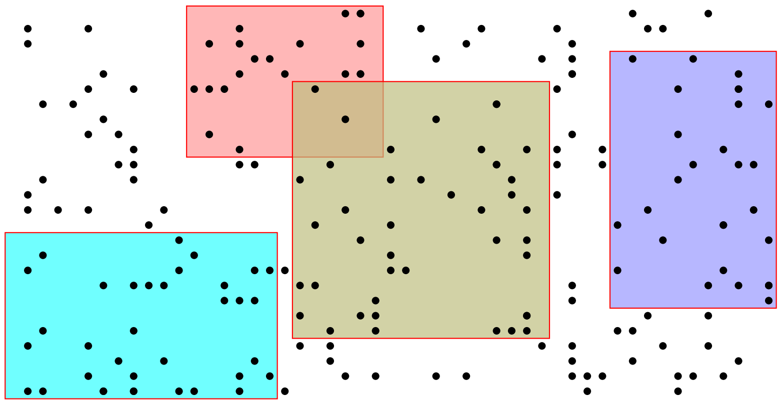

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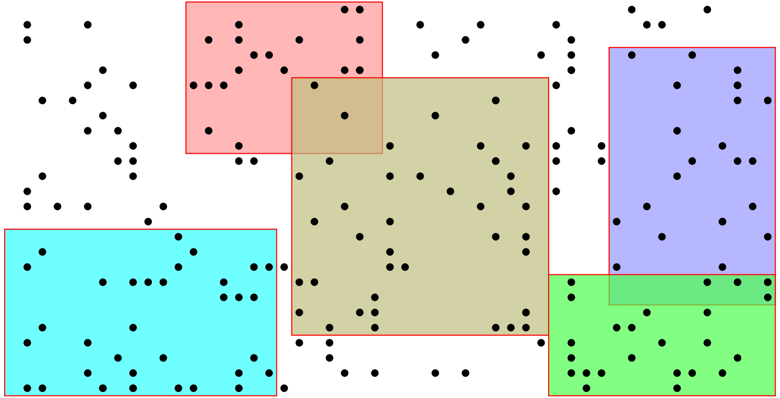

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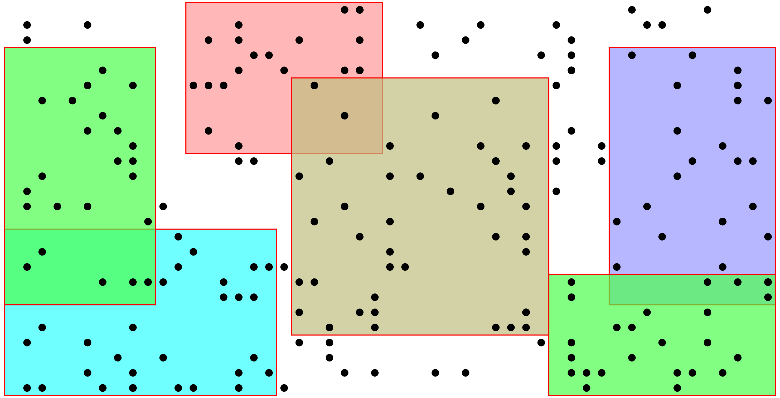

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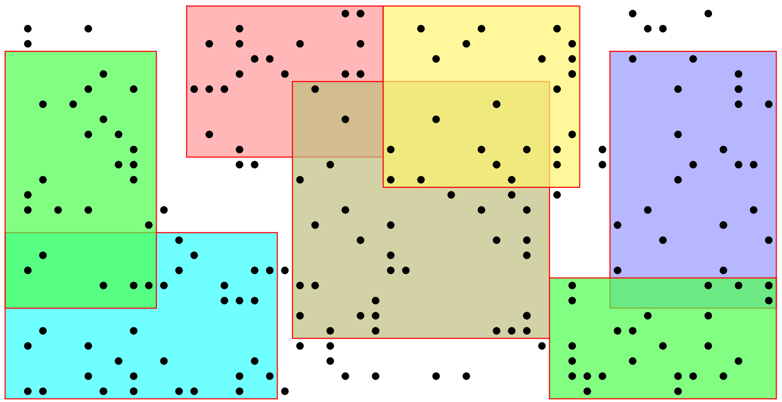

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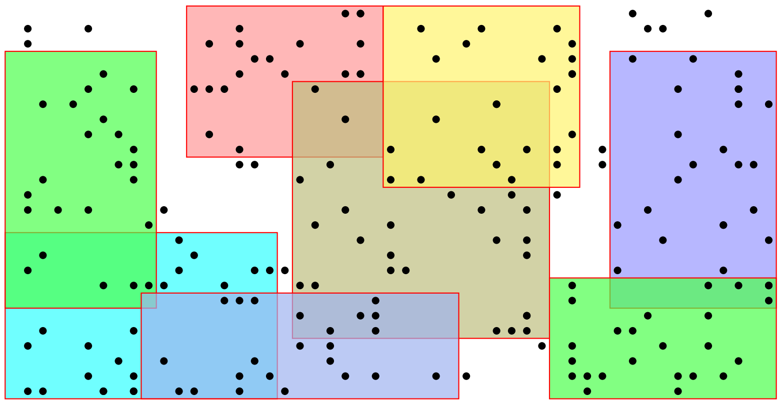

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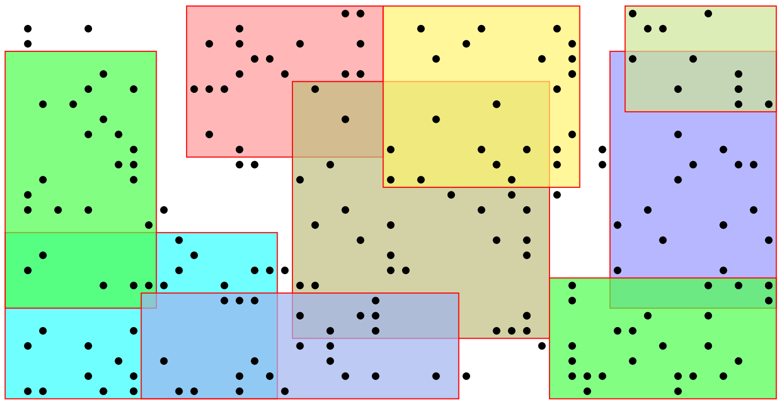

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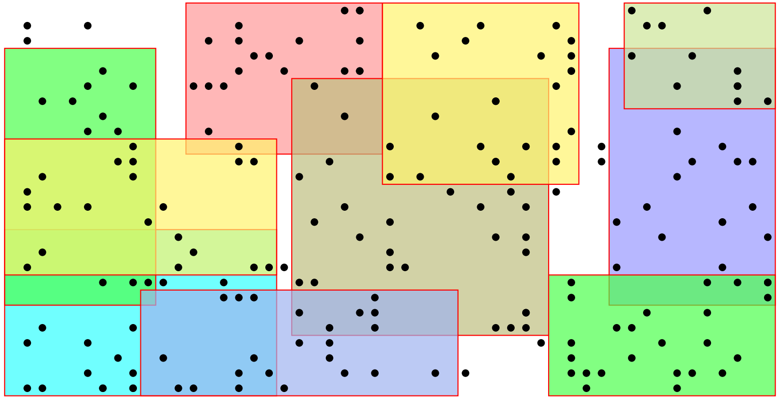

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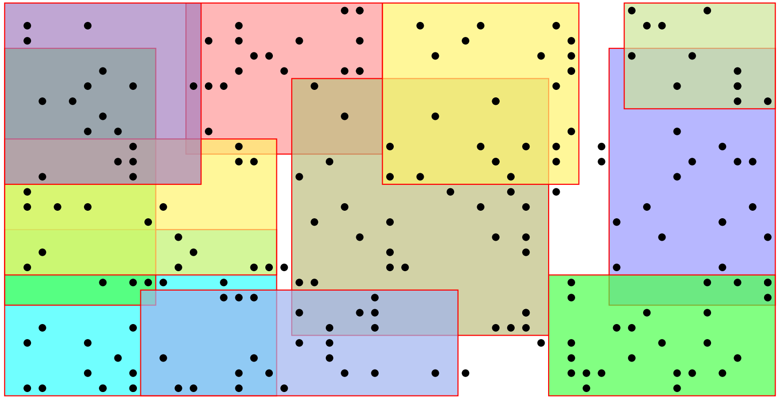

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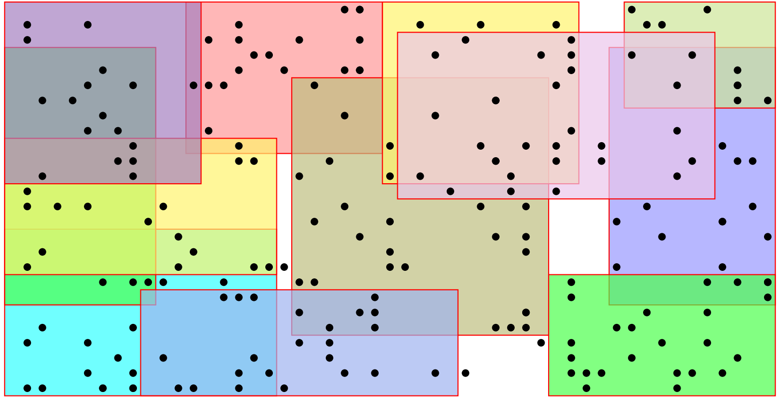

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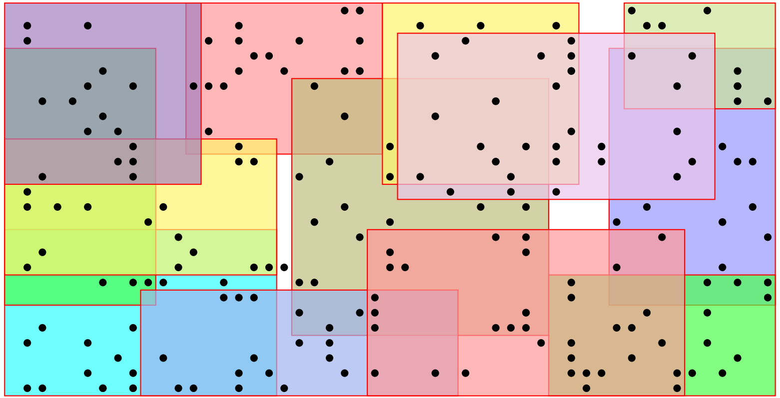

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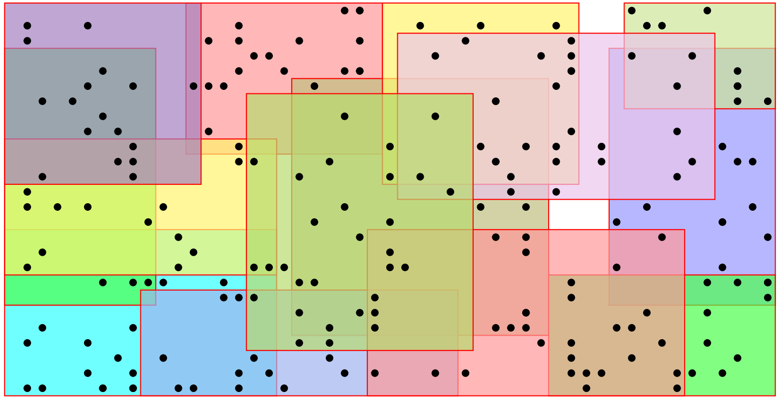

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\text{s.t.} & \forall u \in U & \sum_{i:u\in S_i} x_i & \geq & 1 \\
& \forall i \in \{1,\dots,k\} & x_i & \geq & 0 \\
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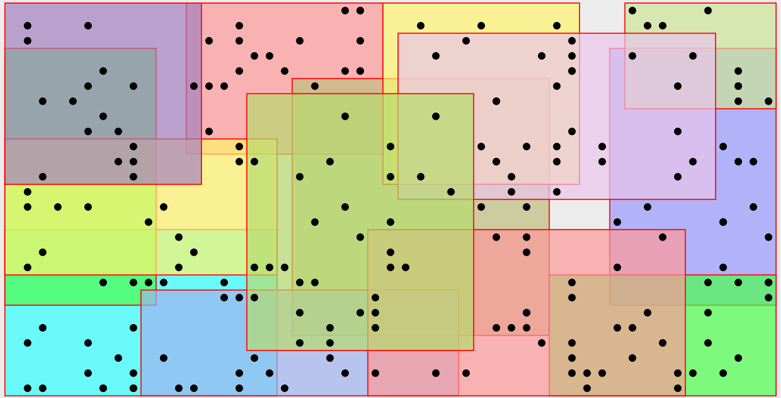

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Given a graph $G = (V, E)$ and a weight w_v for every node. Find a vertex subset $S \subseteq V$ of minimum weight such that every edge is incident to at least one vertex in S.

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Given a set of items $\{1, \ldots, n\}$, where the i-th item has weight w_i and profit p_i, and given a threshold K. Find a subset $I \subseteq \{1, \ldots, n\}$ of items of total weight at most K such that the profit is maximized.

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A linear program LP is a relaxation of an integer program IP if any feasible solution for IP is also feasible for LP and if the objective values of these solutions are identical in both programs.

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Relations

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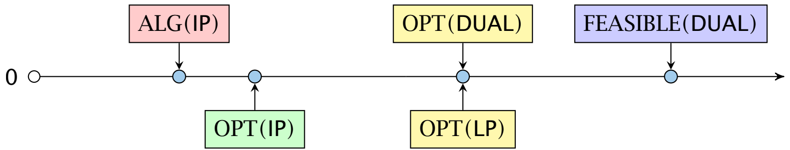

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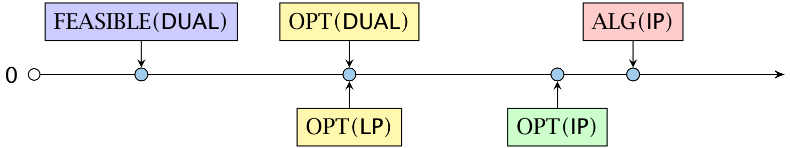

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