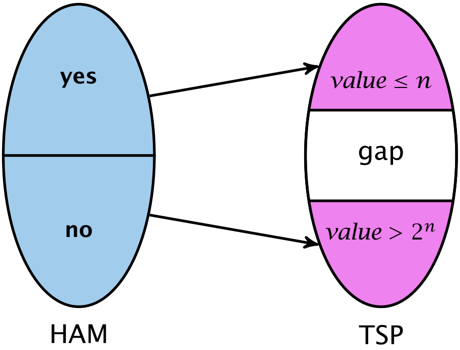
Gap Introducing Reduction



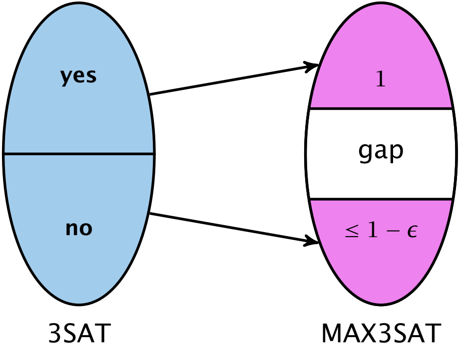
Reduction from Hamiltonian cycle to TSP

- instance that has Hamiltonian cycle is mapped to TSP instance with small cost
- otherwise it is mapped to instance with large cost
- $\Longrightarrow$ there is no $2^n/n$-approximation for TSP

PCP theorem: Approximation View

Theorem 2 (PCP Theorem A)

There exists $\epsilon > 0$ for which there is gap introducing reduction between 3SAT and MAX3SAT.



PCP theorem: Proof System View

Definition 3 (NP)

A language $L \in \mathrm{NP}$ if there exists a polynomial time, deterministic verifier V (a Turing machine), s.t.

[$x \in L$] completeness
There exists a proof string y, $|y| = \mathrm{poly}(|x|)$, s.t. $V(x, y) =$ "accept".

[$x \notin L$] soundness
For any proof string y, $V(x, y) =$ "reject".

Note that requiring $|y| = \mathrm{poly}(|x|)$ for $x \notin L$ does not make a difference (**why?**).

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Probabilistic Checkable Proofs

An Oracle Turing Machine M is a Turing machine that has access to an oracle.

Such an oracle allows M to solve some problem in a single step.

For example having access to a TSP-oracle π_{TSP} would allow M to write a TSP-instance x on a special oracle tape and obtain the answer (yes or no) in a single step.

For such TMs one looks in addition to running time also at query complexity, i.e., how often the machine queries the oracle.

For a proof string y, π_y is an oracle that upon given an index i returns the i-th character y_i of y.

Probabilistic Checkable Proofs

Definition 5 (PCP)

A language $L \in \mathrm{PCP}_{c(n),s(n)}(r(n), q(n))$ if there exists a polynomial time, non-adaptive, randomized verifier V, s.t.

[$x \in L$] There exists a proof string y, s.t. $V^{\pi_y}(x) =$ "accept" with proability $\geq c(n)$.

[$x \notin L$] For any proof string y, $V^{\pi_y}(x) =$ "accept" with probability $\leq s(n)$.

The verifier uses at most $\mathcal{O}(r(n))$ random bits and makes at most $\mathcal{O}(q(n))$ oracle queries.

Probabilistic Checkable Proofs

$c(n)$ is called the completeness. If not specified otw. $c(n) = 1$. Probability of accepting a correct proof.

$s(n) < c(n)$ is called the soundness. If not specified otw. $s(n) = 1/2$. Probability of accepting a wrong proof.

$r(n)$ is called the randomness complexity, i.e., how many random bits the (randomized) verifier uses.

$q(n)$ is the query complexity of the verifier.

Probabilistic Checkable Proofs

- $\mathrm{P} = \mathrm{PCP}(0, 0)$
  verifier without randomness and proof access is deterministic algorithm
- $\mathrm{PCP}(\log n, 0) \subseteq \mathrm{P}$
  we can simulate $\mathcal{O}(\log n)$ random bits in deterministic, polynomial time
- $\mathrm{PCP}(0, \log n) \subseteq \mathrm{P}$
  we can simulate short proofs in polynomial time
- $\mathrm{PCP}(\mathrm{poly}(n), 0) = \mathrm{coRP} \stackrel{?!}{=} \mathrm{P}$
  by definition; coRP is randomized polytime with one sided error (positive probability of accepting NO-instance)

Note that the first three statements also hold with equality

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PCP theorem: Proof System View

Theorem 6 (PCP Theorem B)

$\mathrm{NP} = \mathrm{PCP}(\log n, 1)$

Probabilistic Proof for Graph NonIsomorphism

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Verifier gets input (G_0, G_1) (two graphs with n-nodes)

It expects a proof of the following form:

- For any labeled n-node graph H the H's bit $P[H]$ of the proof fulfills

$$
\begin{aligned}
G_0 \equiv H &\implies P[H] = 0 \\
G_1 \equiv H &\implies P[H] = 1 \\
G_0, G_1 \not\equiv H &\implies P[H] = \text{arbitrary}
\end{aligned}
$$

Probabilistic Proof for Graph NonIsomorphism

Verifier:

- choose $b \in \{0, 1\}$ at random
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If $G_0 \not\equiv G_1$ then by using the obvious proof the verifier will always accept.

If $G_0 \equiv G_1$ a proof only accepts with probability $1/2$.

- suppose $\pi(G_0) = G_1$
- if we accept for $b = 1$ and permutation π_{rand} we reject for $b = 0$ and permutation $\pi_{\text{rand}} \circ \pi$

Version B $\Longrightarrow$ Version A

- For 3SAT there exists a verifier that uses $c \log n$ random bits, reads $q = \mathcal{O}(1)$ bits from the proof, has completeness 1 and soundness $1/2$.
- fix x and r:

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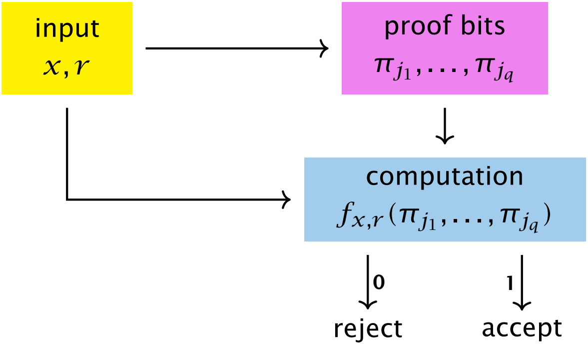

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- transform Boolean formula $f_{x,r}$ into 3SAT formula $C_{x,r}$ (constant size, variables are proof bits)
- consider 3SAT formula $C_x := \bigwedge_r C_{x,r}$

$[x \in L]$ There exists proof string y, s.t. all formulas $C_{x,r}$ evaluate to 1. Hence, all clauses in C_x satisfied.

$[x \notin L]$ For any proof string y, at most 50% of formulas $C_{x,r}$ evaluate to 1. Since each contains only a constant number of clauses, a constant fraction of clauses in C_x are not satisfied.

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Version A $\Longrightarrow$ Version B

We show: Version A $\Longrightarrow \mathrm{NP} \subseteq \mathrm{PCP}_{1,1-\epsilon}(\log n, 1)$.

given $L \in \mathrm{NP}$ we build a PCP-verifier for L

Verifier:

- 3SAT is NP-complete; map instance x for L into 3SAT instance I_x, s.t. I_x satisfiable iff $x \in L$
- map I_x to MAX3SAT instance C_x
- interpret proof as assignment to variables in C_x
- choose random clause X from C_x
- query variable assignment σ for X;
- accept if $X(\sigma) =$ true otw. reject

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[$x \in L$] There exists proof string y, s.t. all clauses in C_x evaluate to 1. In this case the verifier returns 1.

[$x \notin L$] For any proof string y, at most a $(1-\epsilon)$-fraction of clauses in C_x evaluate to 1. The verifier will reject with probability at least ϵ.

To show Theorem B we only need to run this verifier a constant number of times to push rejection probability above $1/2$.

NP ⊆ PCP(poly(n), 1)

PCP(poly(n), 1) means we have a potentially exponentially long proof but we only read a constant number of bits from it.

The idea is to encode an NP-witness (e.g. a satisfying assignment (say n bits)) by a code whose code-words have 2^n bits.

A wrong proof is either

- a code-word whose pre-image does not correspond to a satisfying assignment
- or, a sequence of bits that does not correspond to a code-word

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The Code

$u \in \{0,1\}^n$ (satisfying assignment)

Walsh-Hadamard Code:
$\mathrm{WH}_u : \{0,1\}^n \to \{0,1\}, x \mapsto x^T u$ (over $\mathrm{GF}(2)$)

The code-word for u is WH_u. We identify this function by a bit-vector of length 2^n.

The Code

Lemma 7
If $u \neq u'$ then WH_u and $\mathrm{WH}_{u'}$ differ in at least 2^{n-1} bits.

Proof:
Suppose that $u - u' \neq 0$. Then

$$\mathrm{WH}_u(x) \neq \mathrm{WH}_{u'}(x) \iff (u - u')^T x \neq 0$$

This holds for 2^{n-1} different vectors x.

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Since the set of codewords is the set of all linear functions $\{0,1\}^n$ to $\{0,1\}$ we can check

$$f(x + y) = f(x) + f(y)$$

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Can we just check a constant number of positions?

NP ⊆ PCP(poly(n), 1)

Definition 8

Let $\rho \in [0,1]$. We say that $f, g : \{0,1\}^n \to \{0,1\}$ are ρ-close if

$$\Pr_{x \in \{0,1\}^n}[f(x) = g(x)] \geq \rho \ .$$

Theorem 9 (proof deferred)

Let $f : \{0,1\}^n \to \{0,1\}$ with

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Then there is a linear function $\tilde{f}$ such that f and $\tilde{f}$ are ρ-close.

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Then there is a linear function $\tilde{f}$ *such that* f *and* $\tilde{f}$ *are* ρ*-close.*

NP $\subseteq$ PCP(poly(n), 1)

We need $\mathcal{O}(1/\delta)$ trials to be sure that f is $(1-\delta)$-close to a linear function with (arbitrary) constant probability.

NP ⊆ PCP(poly(n), 1)

Suppose for $\delta < 1/4$ f is $(1-\delta)$-close to some linear function $\tilde{f}$.

$\tilde{f}$ is uniquely defined by f, since linear functions differ on at least half their inputs.

Suppose we are given $x \in \{0,1\}^n$ and access to f. Can we compute $\tilde{f}(x)$ using only constant number of queries?

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1. Choose $x' \in \{0,1\}^n$ u.a.r.
2. Set $x'' := x + x'$.
3. Let $y' = f(x')$ and $y'' = f(x'')$.
4. Output $y' + y''$.

x' and x'' are uniformly distributed (albeit dependent). With probability at least $1 - 2\delta$ we have $f(x') = \tilde{f}(x')$ and $f(x'') = \tilde{f}(x'')$.

Then the above routine returns $\tilde{f}(x)$.

This technique is known as local decoding of the Walsh-Hadamard code.

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$\mathrm{NP} \subseteq \mathrm{PCP}(\mathrm{poly}(n), 1)$

We show that $\mathrm{QUADEQ} \in \mathrm{PCP}(\mathrm{poly}(n), 1)$. The theorem follows since any PCP-class is closed under polynomial time reductions.

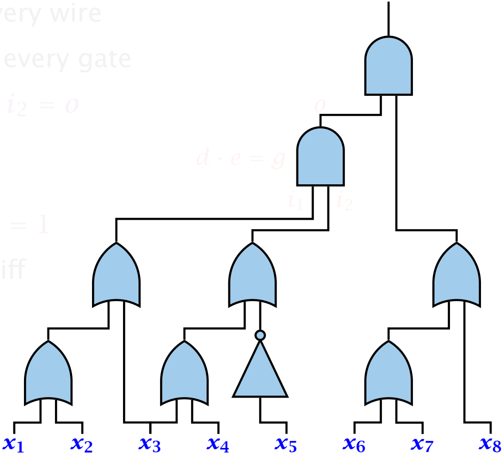
QUADEQ

Given a system of quadratic equations over $\mathrm{GF}(2)$. Is there a solution?

QUADEQ is NP-complete

- given 3SAT instance C represent it as Boolean circuit
  e.g. $C = (x_1 \vee x_2 \vee x_3) \wedge (x_3 \vee x_4 \vee \bar{x}_5) \wedge (x_6 \vee x_7 \vee x_8)$
- add variable for every wire
- add constraint for every gate

  OR: $i_1 + i_2 + i_1 \cdot i_2 = o$
  AND: $i_1 \cdot i_2 = o$
  NEG: $i = 1 - o$
- add constraint $out = 1$
- system is feasible iff
  C is satisfiable



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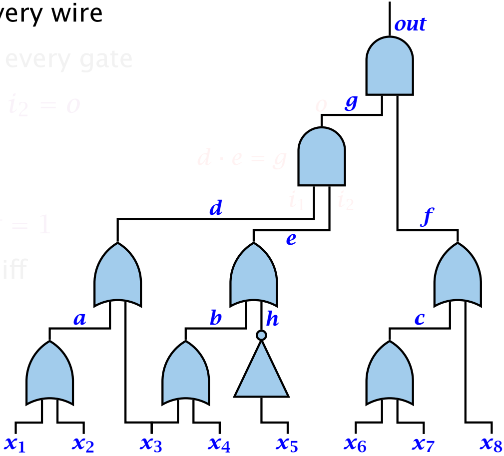

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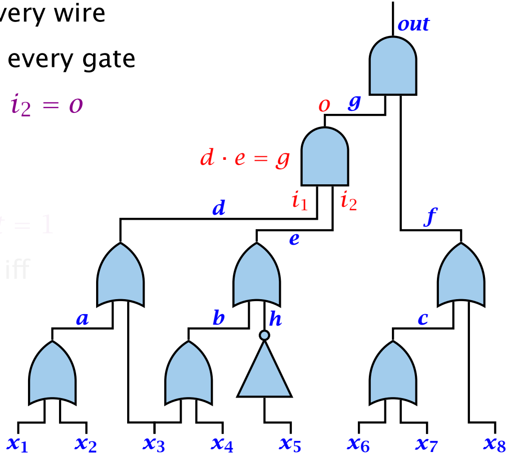

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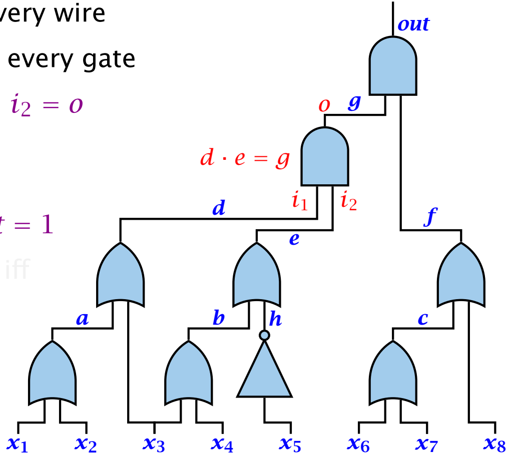

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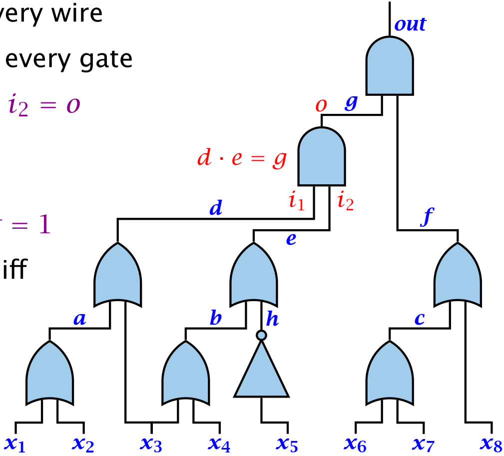

$\mathbf{NP} \subseteq \mathbf{PCP}(\mathbf{poly}(n), 1)$

We encode an instance of QUADEQ by a matrix A that has n^2 columns; one for every pair i, j; and a right hand side vector b.

For an n-dimensional vector x we use $x \otimes x$ to denote the n^2-dimensional vector whose i, j-th entry is $x_i x_j$.

Then we are asked whether

$$A(x \otimes x) = b$$

has a solution.

$\mathbf{NP} \subseteq \mathbf{PCP}(\mathbf{poly}(n), 1)$

Let A, b be an instance of QUADEQ. Let u be a satisfying assignment.

The correct PCP-proof will be the Walsh-Hadamard encodings of u and $u \otimes u$. The verifier will accept such a proof with probability 1.

We have to make sure that we reject proofs that do not correspond to codewords for vectors of the form u, and $u \otimes u$.

We also have to reject proofs that correspond to codewords for vectors of the form z, and $z \otimes z$, where z is not a satisfying assignment.

NP $\subseteq$ PCP(poly(n), 1)

Step 1. Linearity Test.
The proof contains $2^n + 2^{n^2}$ bits. This is interpreted as a pair of functions $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^{n^2} \to \{0,1\}$.

We do a 0.999-linearity test for both functions (requires a constant number of queries).

We also assume that for the remaining constant number of accesses WH-decoding succeeds and we recover $\tilde{f}(x)$.

Hence, our proof will only ever see $\tilde{f}$. To simplify notation we use f for $\tilde{f}$, in the following (similar for g, $\tilde{g}$).

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Step 2. Verify that g encodes $u \otimes u$ where u is string encoded by f.

$f(r) = u^T r$ and $g(z) = w^T z$ since f, g are linear.

- choose r, r' independently, u.a.r. from $\{0,1\}^n$
- if $f(r)f(r') \neq g(r \otimes r')$ reject
- repeat 3 times

$\mathbf{NP} \subseteq \mathbf{PCP}(\mathbf{poly}(n), 1)$

A correct proof survives the test

$$f(r) \cdot f(r')$$

NP $\subseteq$ PCP(poly(n), 1)

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f(r) \cdot f(r') &= u^T r \cdot u^T r' \\
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If $U \neq W$ then $Wr' \neq Ur'$ with probability at least 1/2. Then $r^T W r' \neq r^T U r'$ with probability at least 1/4.

NP ⊆ PCP(poly(n), 1)

Step 3. Verify that f encodes satisfying assignment.

We need to check

$$A_k(u \otimes u) = b_k$$

where A_k is the k-th row of the constraint matrix. But the left hand side is just $g(A_k^T)$.

We can handle this by a single query but checking all constraints would take $\mathcal{O}(m)$ steps.

We compute $r^T A$, where $r \in_R \{0,1\}^m$. If u is not a satisfying assignment then with probability 1/2 the vector r will hit an odd number of violated constraints.

In this case $r^T A(u \otimes u) \neq r^T b_k$. The left hand side is equal to $g(A^T r)$.

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We used the following theorem for the linearity test:

Theorem 9
Let $f : \{0,1\}^n \to \{0,1\}$ *with*

$$\Pr_{x,y\in\{0,1\}^n}\Big[f(x)+f(y)=f(x+y)\Big] \geq \rho > \frac{1}{2} \ .$$

Then there is a linear function $\tilde{f}$ *such that* f *and* $\tilde{f}$ *are* ρ*-close.*

Fourier Transform over $\mathrm{GF}(2)$

In the following we use $\{-1, 1\}$ instead of $\{0, 1\}$. We map $b \in \{0, 1\}$ to $(-1)^b$.

This turns summation into multiplication.

The set of function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ form a 2^n-dimensional Hilbert space.

$\mathbf{NP} \subseteq \mathbf{PCP}(\mathbf{poly}(n), 1)$

Hilbert space

- addition $(f+g)(x) = f(x) + g(x)$
- scalar multiplication $(\alpha f)(x) = \alpha f(x)$
- inner product $\langle f, g \rangle = E_{x \in \{-1,1\}^n}[f(x)g(x)]$
  (bilinear, $\langle f, f \rangle \geq 0$, and $\langle f, f \rangle = 0 \Rightarrow f = 0$)
- completeness: any sequence x_k of vectors for which

$$\sum_{k=1}^{\infty} \|x_k\| < \infty \text{ fulfills } \left\| L - \sum_{k=1}^{N} x_k \right\| \to 0$$

for some vector L.

$\mathrm{NP} \subseteq \mathrm{PCP}(\mathrm{poly}(n), 1)$

standard basis

$$e_x(y) = \begin{cases} 1 & x = y \\ 0 & \text{otw.} \end{cases}$$

Then, $f(x) = \sum_i \alpha_i e_i(x)$ where $\alpha_x = f(x)$, this means the functions e_i form a basis. This basis is orthonormal.

$\mathbf{NP} \subseteq \mathbf{PCP}(\mathbf{poly}(n), 1)$

fourier basis

For $\alpha \subseteq [n]$ define

$$\chi_\alpha(x) = \prod_{i \in \alpha} x_i$$

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This means the χ_α's also define an orthonormal basis. (since we have 2^n orthonormal vectors...)

$\mathbf{NP} \subseteq \mathbf{PCP}(\mathbf{poly}(n), 1)$

A function χ_α multiplies a set of x_i's. Back in the $\mathrm{GF}(2)$-world this means summing a set of z_i's where $x_i = (-1)^{z_i}$.

This means the function χ_α correspond to linear functions in the $\mathrm{GF}(2)$ world.

$\mathbf{NP} \subseteq \mathbf{PCP}(\mathbf{poly}(n), 1)$

We can write any function $f : \{-1,1\}^n \to \mathbb{R}$ as

$$f = \sum_\alpha \hat{f}_\alpha \chi_\alpha$$

We call $\hat{f}_\alpha$ the α^{th} Fourier coefficient.

Lemma 10

1. $\langle f, g\rangle = \sum_\alpha f_\alpha g_\alpha$
2. $\langle f, f\rangle = \sum_\alpha f_\alpha^2$

Note that for Boolean functions $f : \{-1,1\}^n \to \{-1,1\}$, $\langle f, f\rangle = 1$.

Linearity Test

in GF(2):

We want to show that if $\Pr_{x,y}[f(x) + f(y) = f(x + y)]$ is large than f has a large agreement with a linear function.

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in Hilbert space: (we will prove)
Suppose $f : \{\pm 1\}^n \to \{-1, 1\}$ fulfills

$$\Pr_{x,y}[f(x)f(y) = f(x \circ y)] \geq \frac{1}{2} + \epsilon \ .$$

Then there is some $\alpha \subseteq [n]$, s.t. $\hat{f}_\alpha \geq 2\epsilon$.

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$$2\epsilon \leq \hat{f}_\alpha = \langle f, \chi_\alpha\rangle = \text{agree} - \text{disagree} = 2\text{agree} - 1$$

This gives that the agreement between f and χ_α is at least $\frac{1}{2} + \epsilon$.

Linearity Test

$$\Pr_{x,y}[f(x \circ y) = f(x)f(y)] \geq \frac{1}{2} + \epsilon$$

means that the fraction of inputs x, y on which $f(x \circ y)$ and $f(x)f(y)$ agree is at least $1/2 + \epsilon$.

This gives

$$\begin{aligned} E_{x,y}[f(x \circ y)f(x)f(y)] &= \text{agreement} - \text{disagreement} \\ &= 2\text{agreement} - 1 \\ &\geq 2\epsilon \end{aligned}$$

$$2\epsilon \leq E_{x,y}\left[f(x \circ y)f(x)f(y)\right]$$

$$\begin{aligned}2\epsilon &\le E_{x,y}\Big[f(x\circ y)f(x)f(y)\Big]\\ &= E_{x,y}\Big[\Big(\sum_\alpha \hat{f}_\alpha\chi_\alpha(x\circ y)\Big)\cdot\Big(\sum_\beta \hat{f}_\beta\chi_\beta(x)\Big)\cdot\Big(\sum_\gamma \hat{f}_\gamma\chi_\gamma(y)\Big)\Big]\end{aligned}$$

$$
\begin{aligned}
2\epsilon &\le E_{x,y}\left[f(x\circ y)f(x)f(y) \right] \\
&= E_{x,y}\left[\left(\sum_\alpha \hat{f}_\alpha \chi_\alpha(x\circ y)\right)\cdot\left(\sum_\beta \hat{f}_\beta \chi_\beta(x)\right)\cdot\left(\sum_\gamma \hat{f}_\gamma \chi_\gamma(y)\right) \right] \\
&= E_{x,y}\left[\sum_{\alpha,\beta,\gamma} \hat{f}_\alpha \hat{f}_\beta \hat{f}_\gamma \chi_\alpha(x)\chi_\alpha(y)\chi_\beta(x)\chi_\gamma(y) \right]
\end{aligned}
$$

$$
\begin{aligned}
2\epsilon &\le E_{x,y}\Big[f(x \circ y) f(x) f(y)\Big] \\
&= E_{x,y}\Big[\Big(\sum_\alpha \hat{f}_\alpha \chi_\alpha(x \circ y)\Big) \cdot \Big(\sum_\beta \hat{f}_\beta \chi_\beta(x)\Big) \cdot \Big(\sum_\gamma \hat{f}_\gamma \chi_\gamma(y)\Big)\Big] \\
&= E_{x,y}\Big[\sum_{\alpha,\beta,\gamma} \hat{f}_\alpha \hat{f}_\beta \hat{f}_\gamma \chi_\alpha(x)\chi_\alpha(y)\chi_\beta(x)\chi_\gamma(y)\Big] \\
&= \sum_{\alpha,\beta,\gamma} \hat{f}_\alpha \hat{f}_\beta \hat{f}_\gamma \cdot E_x\Big[\chi_\alpha(x)\chi_\beta(x)\Big] E_y\Big[\chi_\alpha(y)\chi_\gamma(y)\Big]
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&= E_{x,y}\Big[\sum\nolimits_{\alpha,\beta,\gamma} \hat{f}_{\alpha}\hat{f}_{\beta}\hat{f}_{\gamma}\chi_{\alpha}(x)\chi_{\alpha}(y)\chi_{\beta}(x)\chi_{\gamma}(y)\Big] \\
&= \sum\nolimits_{\alpha,\beta,\gamma} \hat{f}_{\alpha}\hat{f}_{\beta}\hat{f}_{\gamma} \cdot E_x\Big[\chi_{\alpha}(x)\chi_{\beta}(x)\Big] E_y\Big[\chi_{\alpha}(y)\chi_{\gamma}(y)\Big] \\
&= \sum_{\alpha} \hat{f}_{\alpha}^{3}
\end{aligned}
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&= \sum\nolimits_{\alpha,\beta,\gamma}\hat{f}_{\alpha}\hat{f}_{\beta}\hat{f}_{\gamma}\cdot E_x\Big[\chi_{\alpha}(x)\chi_{\beta}(x)\Big]E_y\Big[\chi_{\alpha}(y)\chi_{\gamma}(y)\Big]\\
&= \sum_{\alpha}\hat{f}_{\alpha}^3\\
&\le \max_{\alpha}\hat{f}_{\alpha}\cdot\sum_{\alpha}\hat{f}_{\alpha}^2 = \max_{\alpha}\hat{f}_{\alpha}
\end{aligned}
$$

Approximation Preserving Reductions

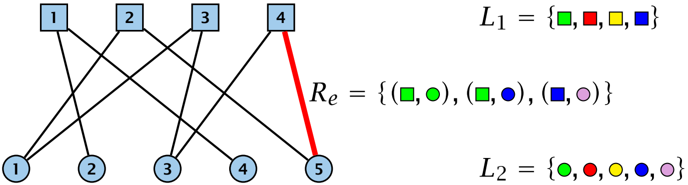
AP-reduction

- $x \in I_1 \Rightarrow f(x,r) \in I_2$
- $\mathrm{SOL}_1(x) \neq \emptyset \Rightarrow \mathrm{SOL}_1(f(x,r)) \neq \emptyset$
- $y \in \mathrm{SOL}_2(f(x,r)) \Rightarrow g(x,y,r) \in \mathrm{SOL}_1(x)$
- f, g are polynomial time computable
- $R_2(f(x,r),y) \leq r \Rightarrow R_1(x, g(x,y,r)) \leq 1 + \alpha(r-1)$

Label Cover

Input:

- bipartite graph $G = (V_1, V_2, E)$
- label sets L_1, L_2
- for every edge $(u, v) \in E$ a relation $R_{u,v} \subseteq L_1 \times L_2$ that describe assignments that make the edge happy.
- maximize number of happy edges



Label Cover

- an instance of label cover is (d_1, d_2)-regular if every vertex in L_1 has degree d_1 and every vertex in L_2 has degree d_2.
- if every vertex has the same degree d the instance is called d-regular

Minimization version:

- assign a set $L_x \subseteq L_1$ of labels to every node $x \in L_1$ and a set $L_y \subseteq L_2$ to every node $y \in L_2$
- make sure that for every edge (x, y) there is $\ell_x \in L_x$ and $\ell_y \in L_y$ s.t. $(\ell_x, \ell_y) \in R_{x,y}$
- minimize $\sum_{x \in L_1} |L_x| + \sum_{y \in L_2} |L_y|$ (total labels used)

MAX E3SAT via Label Cover

instance:

$\Phi(x) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_4 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_4)$

corresponding graph:

label sets: $L_1 = \{T, F\}^3, L_2 = \{T, F\}$ (T=true, F=false)

relation: $R_{C,x_i} = \{((u_i, u_j, u_k), u_i)\}$, where the clause C is over variables x_i, x_j, x_k and assignment (u_i, u_j, u_k) satisfies C

$$R = \{((F,F,F),F), ((F,T,F),F), ((F,F,T),T), ((F,T,T),T),$$
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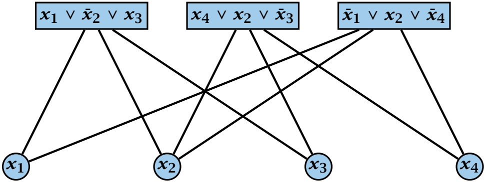

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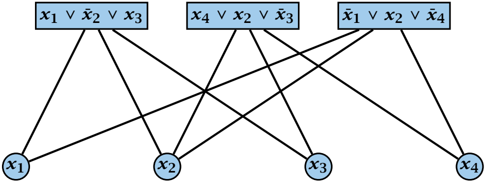

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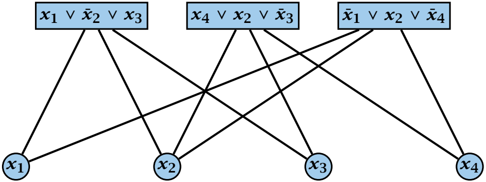

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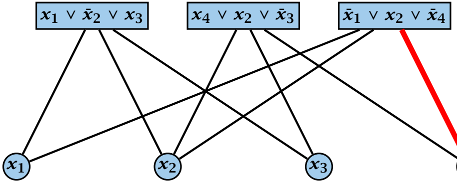

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MAX E3SAT via Label Cover

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If we can satisfy k out of m clauses in ϕ we can make at least $3k + 2(m-k)$ edges happy.

MAX E3SAT via Label Cover

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Proof:

- for V_2 use the setting of the assignment that satisfies k clauses
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- for unsatisfied clauses flip assignment of one of the variables; this makes one incident edge unhappy (gives $2(m-k)$ happy edges)

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MAX E3SAT via Label Cover

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If we can satisfy at most k clauses in Φ we can make at most $3k + 2(m - k) = 2m + k$ edges happy.

Proof:

- the labeling of nodes in V_2 gives an assignment
- every unsatisfied clause in this assignment cannot be assigned a label that satisfies all 3 incident edges
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Hardness for Label Cover

We cannot distinguish between the following two cases

- all $3m$ edges can be made happy
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$(3,5)$-regular instances

Theorem 13
There is a constant ρ s.t. MAXE3SAT is hard to approximate with a factor of ρ even if restricted to instances where a variable appears in exactly 5 clauses.

Then our reduction has the following properties:

- the resulting Label Cover instance is $(3,5)$-regular
- it is hard to approximate for a constant $\alpha < 1$
- given a label ℓ_1 for x there is at most one label ℓ_2 for y that makes edge (x,y) happy (uniqueness property)

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$(3,5)$-regular instances

The previous theorem can be obtained with a series of gap-preserving reductions:

- MAX3SAT $\leq$ MAX3SAT(≤ 29)
- MAX3SAT(≤ 29) $\leq$ MAX3SAT(≤ 5)
- MAX3SAT(≤ 5) $\leq$ MAX3SAT$(= 5)$
- MAX3SAT$(= 5)$ $\leq$ MAXE3SAT$(= 5)$

Here MAX3SAT(≤ 29) is the variant of MAX3SAT in which a variable appears in at most 29 clauses. Similar for the other problems.

Regular instances

Theorem 14
There is a constant $\alpha < 1$ such if there is an α-approximation algorithm for Label Cover on 15-regular instances than P=NP.

Given a label ℓ_1 for $x \in V_1$ there is at most one label ℓ_2 for y that makes (x, y) happy. (uniqueness property)

Parallel Repetition

We would like to increase the inapproximability for Label Cover.

In the verifier view, in order to decrease the acceptance probability of a wrong proof (or as here: a pair of wrong proofs) one could repeat the verification several times.

Unfortunately, we have a 2P1R-system, i.e., we are stuck with a single round and cannot simply repeat.

The idea is to use parallel repetition, i.e., we simply play several rounds in parallel and hope that the acceptance probability of wrong proofs goes down.

Parallel Repetition

Given Label Cover instance I with $G = (V_1, V_2, E)$, label sets L_1 and L_2 we construct a new instance I':

- $V_1' = V_1^k = V_1 \times \cdots \times V_1$
- $V_2' = V_2^k = V_2 \times \cdots \times V_2$
- $L_1' = L_1^k = L_1 \times \cdots \times L_1$
- $L_2' = L_2^k = L_2 \times \cdots \times L_2$
- $E' = E^k = E \times \cdots \times E$

An edge $((x_1, \ldots, x_k), (y_1, \ldots, y_k))$ whose end-points are labelled by $(\ell_1^x, \ldots, \ell_k^x)$ and $(\ell_1^y, \ldots, \ell_k^y)$ is happy if $(\ell_i^x, \ell_i^y) \in R_{x_i, y_i}$ for all i.

Parallel Repetition

If I is regular than also I'.

If I has the uniqueness property than also I'.

Did the gap increase?

- Suppose we have labelling ℓ_1, ℓ_2 that satisfies just an α-fraction of edges in I.
- We transfer this labelling to instance I':
  vertex $(x_1, \dots, x_k)$ gets label $(\ell_1(x_1), \dots, \ell_1(x_k))$,
  vertex $(y_1, \dots, y_k)$ gets label $(\ell_2(y_1), \dots, \ell_2(y_k))$.
- How many edges are happy?

Does this always work?

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  only $(\alpha|E|)^k$ out of $|E|^k$!!! (just an α^k fraction)

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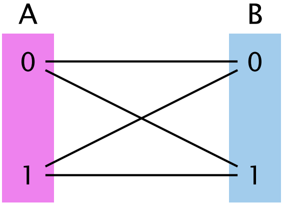
Counter Example

Non interactive agreement:

- Two provers A and B
- The verifier generates two random bits b_A, and b_B, and sends one to A and one to B.
- Each prover has to answer one of A_0, A_1, B_0, B_1 with the meaning $A_0 :=$ prover A has been given a bit with value 0.
- The provers win if they give the same answer and if the answer is correct.

Counter Example

The provers can win with probability at most $1/2$.



Regardless what we do 50% of edges are unhappy!

Counter Example

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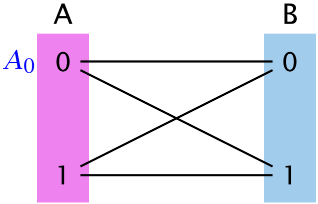

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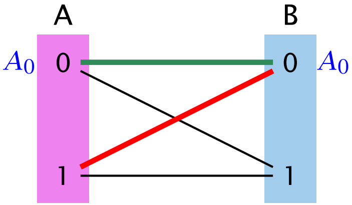

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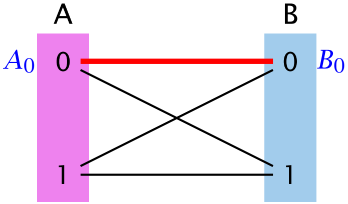

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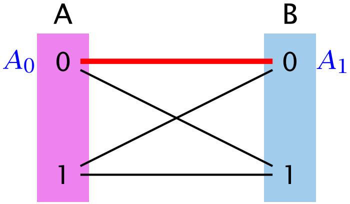

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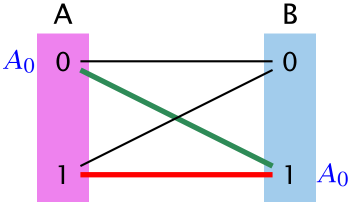

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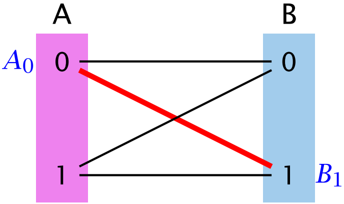

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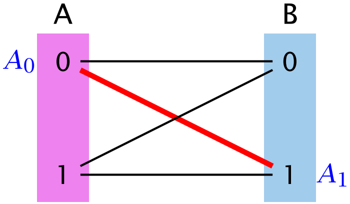

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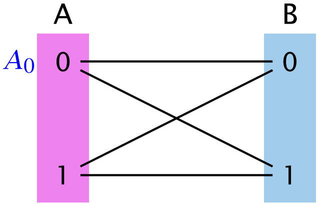

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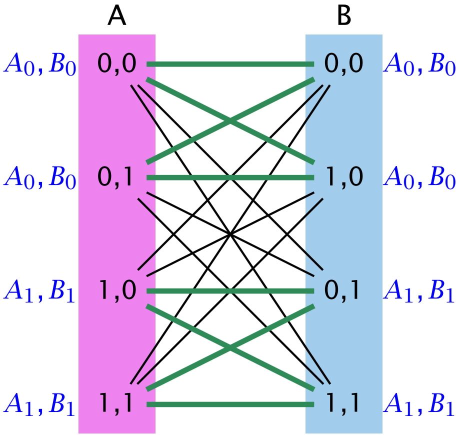
Counter Example

In the repeated game the provers can also win with probability $1/2$:



Boosting

Theorem 15
There is a constant $c > 0$ *such if* $\mathrm{OPT}(I) = |E|(1-\delta)$ *then* $\mathrm{OPT}(I') \leq |E'|(1-\delta)^{\frac{ck}{\log L}}$*, where* $L = |L_1| + |L_2|$ *denotes total number of labels in* I*.*

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Hardness of Label Cover

Theorem 16

There are constants $c > 0$, $\delta < 1$ s.t. for any k we cannot distinguish regular instances for Label Cover in which either

- $\mathrm{OPT}(I) = |E|$, *or*
- $\mathrm{OPT}(I) = |E|(1-\delta)^{ck}$

unless each problem in NP has an algorithm running in time $\mathcal{O}(n^{\mathcal{O}(k)})$.

Corollary 17

There is no α-approximation for Label Cover for any constant α.

Hardness of Set Cover

Theorem 18

There exist regular Label Cover instances s.t. we cannot distinguish whether

- *all edges are satisfiable, or*
- *at most a $1/\log^2(|L_1||E|)$-fraction is satisfiable*

unless NP-problems have algorithms with running time $\mathcal{O}(n^{\mathcal{O}(\log\log n)})$.

choose $k \geq \frac{2}{c}\log_{1/(1-\delta)}(\log(|L_1||E|)) = \mathcal{O}(\log\log n)$.

Hardness of Set Cover

Partition System (s, t, h)

- universe U of size s
- t pairs of sets $(A_1, \bar{A}_1), \dots, (A_t, \bar{A}_t)$;
  $A_i \subseteq U, \bar{A}_i = U \setminus A_i$
- choosing from any h pairs only one of A_i, $\bar{A}_i$ we do not cover the whole set U

we will show later:
for any h, t with $h \le t$ there exist systems with $s = |U| \le 4t^2 2^h$

Hardness of Set Cover

Given a Label Cover instance we construct a Set Cover instance;

The universe is $E \times U$, where U is the universe of some partition system; ($t = |L_1|$, $h = \log(|E||L_1|)$)

for all $u \in V_1, \ell_1 \in L_1$

$$S_{u,\ell_1} = \{((u,v),a) \mid (u,v) \in E, a \in A_{\ell_1}\}$$

for all $v \in V_2, \ell_2 \in L_2$

$S_{v,\ell_2} = \{((u,v),a) \mid (u,v) \in E, a \in \bar{A}_{\ell_1}$, where $(\ell_1,\ell_2) \in R_{(u,v)}\}$

note that S_{v,ℓ_2} is well defined because of uniqueness property

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Hardness of Set Cover

Suppose that we can make all edges happy.

Choose sets S_{u,ℓ_1}'s and S_{v,ℓ_2}'s, where ℓ_1 is the label we assigned to u, and ℓ_2 the label for v. ($|V_1|$+$|V_2|$ sets)

For an edge (u,v), S_{v,ℓ_2} contains $\{(u,v)\} \times A_{\ell_2}$. For a happy edge S_{u,ℓ_1} contains $\{(u,v)\} \times \bar{A}_{\ell_2}$.

Since all edges are happy we have covered the whole universe.

If the Label Cover instance is completely satisfiable we can cover with $|V_1| + |V_2|$ sets.

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Hardness of Set Cover

Lemma 19
Given a solution to the set cover instance using at most $\frac{h}{8}(|V_1| + |V_2|)$ sets we can find a solution to the Label Cover instance satisfying at least $\frac{2}{h^2}|E|$ edges.

If the Label Cover instance cannot satisfy a $2/h^2$-fraction we cannot cover with $\frac{h}{8}(|V_1| + |V_2|)$ sets.

Since differentiating between both cases for the Label Cover instance is hard, we have an $\mathcal{O}(h)$-hardness for Set Cover.

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Hardness of Set Cover

- n_u: number of $S_{u,i}$'s in cover
- n_v: number of $S_{v,j}$'s in cover
- at most 1/4 of the vertices can have $n_u, n_v \geq h/2$; mark these vertices
- at least half of the edges have both end-points unmarked, as the graph is regular
- for such an edge (u, v) we must have chosen $S_{u,i}$ and a corresponding $S_{v,j}$, s.t. $(i, j) \in R_{u,v}$ (making (u, v) happy)
- we choose a random label for u from the (at most $h/2$) chosen $S_{u,i}$-sets and a random label for v from the (at most $h/2$) $S_{v,j}$-sets
- (u, v) gets happy with probability at least $4/h^2$
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Set Cover

Theorem 20
There is no $\frac{1}{32}\log n$-approximation for the unweighted Set Cover problem unless problems in NP can be solved in time $\mathcal{O}(n^{\mathcal{O}(\log\log n)})$.

Given label cover instance (V_1, V_2, E), label sets L_1 and L_2;

Set $h = \log(|E||L_1|)$ and $t = |L_1|$; Size of partition system is

$$s = |U| = 4t^2 2^h = 4|L_1|^2(|E||L_1|)^2 = 4|E|^2|L_1|^4$$

The size of the ground set is then

$$n = |E||U| = 4|E|^3|L_2|^4 \le (|E||L_2|)^4$$

for sufficiently large $|E|$. Then $h \ge \frac{1}{4}\log n$.

If we get an instance where all edges are satisfiable there exists a cover of size only $|V_1| + |V_2|$.

If we find a cover of size at most $\frac{h}{8}(|V_1| + |V_2|)$ we can use this to satisfy at least a fraction of $2/h^2 \ge 1/\log^2(|E||L_1|)$ of the edges. this is not possible...

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$$n = |E||U| = 4|E|^3|L_2|^4 \le (|E||L_2|)^4$$

for sufficiently large $|E|$. Then $h \ge \frac{1}{4}\log n$.

If we get an instance where all edges are satisfiable there exists a cover of size only $|V_1| + |V_2|$.

If we find a cover of size at most $\frac{h}{8}(|V_1| + |V_2|)$ we can use this to satisfy at least a fraction of $2/h^2 \ge 1/\log^2(|E||L_1|)$ of the edges. this is not possible...

Partition Systems

Lemma 21

Given h and t with $h \leq t$, there is a partition system of size $s = \ln(4t)h2^h \leq 4t^2 2^h$.

We pick t sets at random from the possible $2^{|U|}$ subsets of U.

Fix a choice of h of these sets, and a choice of h bits (whether we choose A_i or $\bar{A}_i$). There are $2^h \cdot \binom{t}{h}$ such choices.

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What is the probability that a given choice covers U?

The probability that an element $u \in A_i$ is $1/2$ (same for $\bar{A}_i$).

The probability that u is covered is $1 - \frac{1}{2^h}$.

The probability that all u are covered is $(1 - \frac{1}{2^h})^s$

The probability that there exists a choice such that all u are covered is at most

$$\binom{t}{h} 2^h \left(1 - \frac{1}{2^h}\right)^s \le (2t)^h e^{-s/2^h} = (2t)^h \cdot e^{-h\ln(4t)} < \frac{1}{2} \ .$$

The random process outputs a partition system with constant probability!

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Advanced PCP Theorem

Theorem 22
For any positive constant $\epsilon > 0$, it is the case that $\mathrm{NP} \subseteq \mathrm{PCP}_{1-\epsilon,1/2+\epsilon}(\log n, 3)$. Moreover, the verifier just reads three bits from the proof, and bases its decision only on the parity of these bits.

It is NP-hard to approximate a MAXE3LIN problem by a factor better than $1/2 + \delta$, for any constant δ.

It is NP-hard to approximate MAX3SAT better than $7/8 + \delta$, for any constant δ.