

# Duality

How do we get an upper bound to a maximization LP?

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

Note that a lower bound is easy to derive. Every choice of  $a, b \geq 0$  gives us a lower bound (e.g.  $a = 12, b = 28$  gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the  $i$ -th row with  $y_i \geq 0$ ) such that  $\sum_i y_i a_{ij} \geq c_j$  then  $\sum_i y_i b_i$  will be an upper bound.

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## Definition 2

Let  $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$  be a linear program  $P$  (called the primal linear program).

The linear program  $D$  defined by

$$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

is called the **dual problem**.

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## Lemma 3

*The dual of the dual problem is the primal problem.*

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# Weak Duality

Let  $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$  and  
 $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$  be a primal dual pair.

$x$  is primal feasible iff  $x \in \{x \mid Ax \leq b, x \geq 0\}$

$y$  is dual feasible, iff  $y \in \{y \mid A^T y \geq c, y \geq 0\}$ .

## Theorem 4 (Weak Duality)

*Let  $\hat{x}$  be primal feasible and let  $\hat{y}$  be dual feasible. Then*

$$c^T \hat{x} \leq z \leq w \leq b^T \hat{y} .$$

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$$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \quad (\hat{x} \geq 0)$$

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This gives

$$c^T \hat{x} \leq y^T A \hat{x} \leq b^T y .$$

Since, there exists primal feasible  $\hat{x}$  with  $c^T \hat{x} = z$ , and dual feasible  $\hat{y}$  with  $b^T \hat{y} = w$  we get  $z \leq w$ .

If  $P$  is unbounded then  $D$  is infeasible.

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## 5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

# Proof

**Primal:**

$$\max\{c^T x \mid Ax = b, x \geq 0\}$$

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$$\begin{aligned} \max\{c^T x \mid Ax = b, x \geq 0\} \\ = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \end{aligned}$$

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## Dual:

$$\min\{[b^T \ -b^T]y \mid [A^T \ -A^T]y \geq c, y \geq 0\}$$

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**Dual:**

$$\begin{aligned} & \min\{[b^T \ -b^T]y \mid [A^T \ -A^T]y \geq c, y \geq 0\} \\ &= \min\left\{[b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \mid [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\} \end{aligned}$$

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# Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with **reduced cost**

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to  $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$  is solution to the **dual**  $\min\{b^T y \mid A^T y \geq c\}$ .

Hence, the solution is optimal.

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## 5.3 Strong Duality

$$P = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

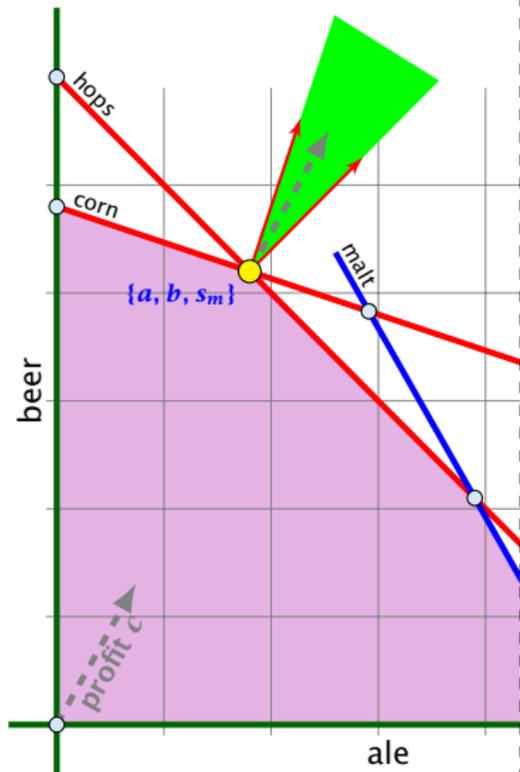
$n_A$ : number of variables,  $m_A$ : number of constraints

We can put the non-negativity constraints into  $A$  (which gives us unrestricted variables):  $\bar{P} = \max\{c^T x \mid \bar{A}x \leq \bar{b}\}$

$$n_{\bar{A}} = n_A, m_{\bar{A}} = m_A + n_A$$

Dual  $D = \min\{\bar{b}^T y \mid \bar{A}^T y = c, y \geq 0\}$ .

## 5.3 Strong Duality



If we have a conic combination  $y$  of  $c$  then  $b^T y$  is an upper bound of the profit we can obtain (**weak duality**):

$$c^T x = (\bar{A}^T y)^T x = y^T \bar{A} x \leq y^T \bar{b}$$

If  $x$  and  $y$  are optimal then the **duality gap** is 0 (**strong duality**). This means

$$\begin{aligned} 0 &= c^T x - y^T \bar{b} \\ &= (\bar{A}^T y)^T x - y^T \bar{b} \\ &= y^T (\bar{A} x - \bar{b}) \end{aligned}$$

The last term can only be 0 if  $y_i$  is 0 whenever the  $i$ -th constraint is not tight. This means we have a conic combination of  $c$  by normals (columns of  $\bar{A}^T$ ) of **tight** constraints.

Conversely, if we have  $x$  such that the normals of tight constraint (at  $x$ ) give rise to a conic combination of  $c$ , we know that  $x$  is optimal.

The profit vector  $c$  lies in the cone generated by the normals for the hops and the corn constraint (the tight constraints).

# Strong Duality

## Theorem 5 (Strong Duality)

Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z^*$  and  $w^*$  denote the optimal solution to  $P$  and  $D$ , respectively.

Then

$$z^* = w^*$$

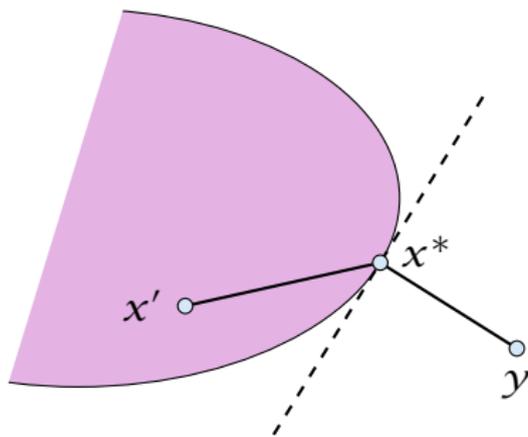
## Lemma 6 (Weierstrass)

Let  $X$  be a compact set and let  $f(x)$  be a continuous function on  $X$ . Then  $\min\{f(x) : x \in X\}$  exists.

**(without proof)**

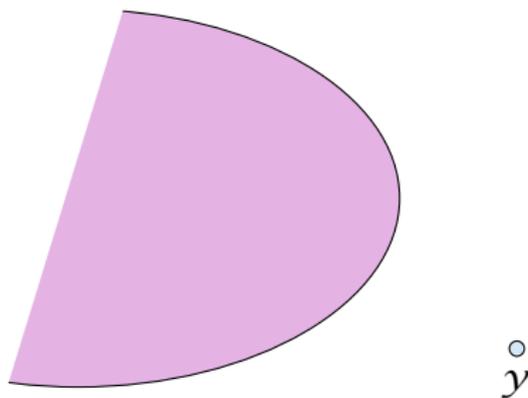
## Lemma 7 (Projection Lemma)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty convex set, and let  $y \notin X$ . Then there exist  $x^* \in X$  with minimum distance from  $y$ . Moreover for all  $x \in X$  we have  $(y - x^*)^T (x - x^*) \leq 0$ .



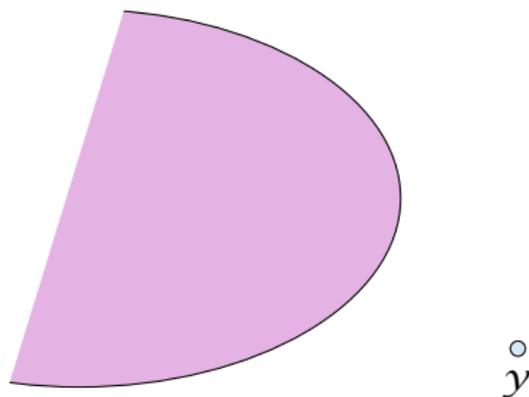
# Proof of the Projection Lemma

- ▶ Define  $f(x) = \|y - x\|$ .
- ▶ We want to apply Weierstrass but  $X$  may not be bounded.
- ▶  $X \neq \emptyset$ . Hence, there exists  $x' \in X$ .
- ▶ Define  $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$ . This set is closed and bounded.
- ▶ Applying Weierstrass gives the existence.



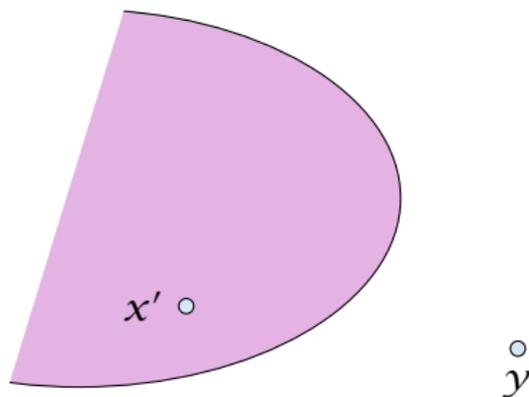
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- ▶ Define  $f(x) = \|y - x\|$ .
- ▶ We want to apply Weierstrass but  $X$  may not be bounded.
- ▶  $X \neq \emptyset$ . Hence, there exists  $x' \in X$ .
- ▶ Define  $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$ . This set is closed and bounded.
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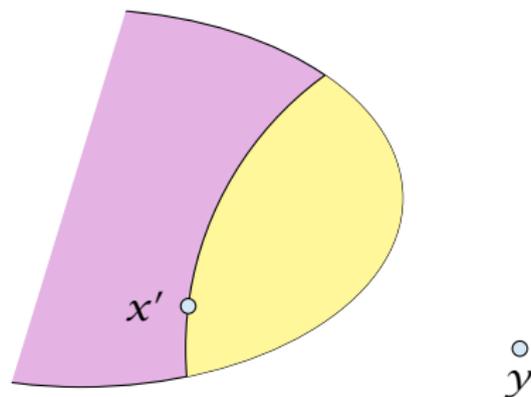
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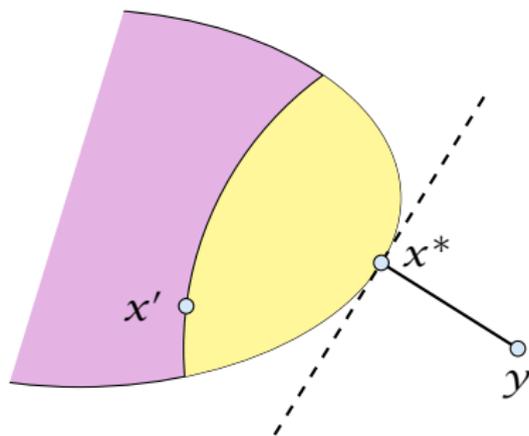
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$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2 \|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$

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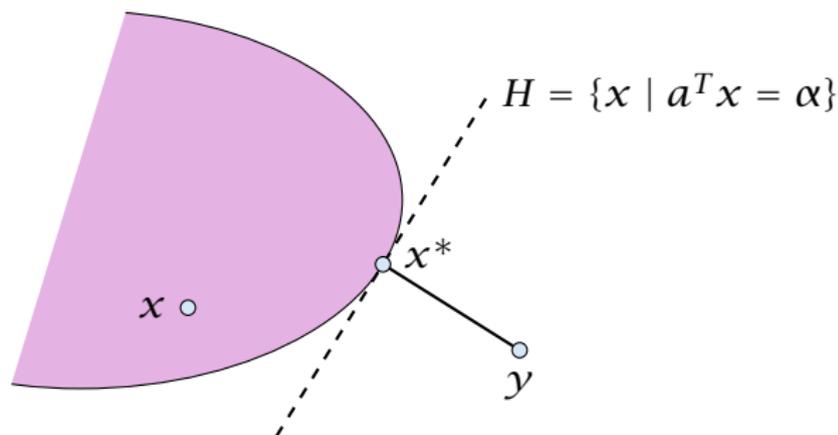
Letting  $\epsilon \rightarrow 0$  gives the result.

## Theorem 8 (Separating Hyperplane)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty closed convex set, and let  $y \notin X$ . Then there exists a *separating hyperplane*  $\{x \in \mathbb{R}^m : a^T x = \alpha\}$  where  $a \in \mathbb{R}^m$ ,  $\alpha \in \mathbb{R}$  that *separates*  $y$  from  $X$ . ( $a^T y < \alpha$ ;  $a^T x \geq \alpha$  for all  $x \in X$ )

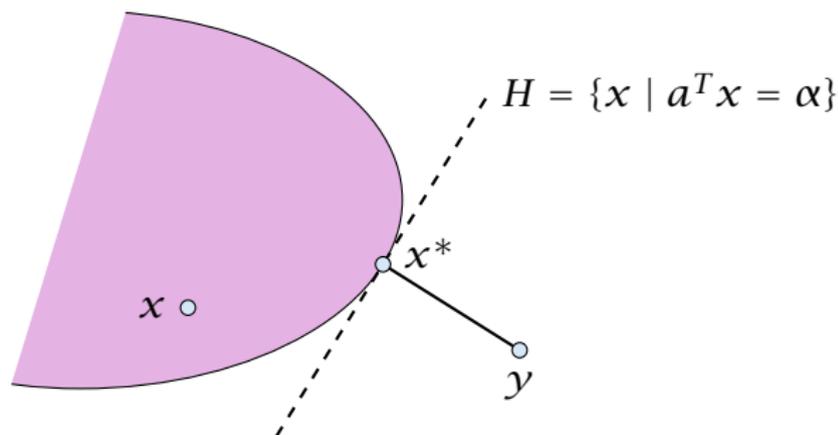
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- ▶ Let  $x^* \in X$  be closest point to  $y$  in  $X$ .
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- ▶ Choose  $a = (x^* - y)$  and  $\alpha = a^T x^*$ .
- ▶ For  $x \in X$ :  $a^T(x - x^*) \geq 0$ , and, hence,  $a^T x \geq \alpha$ .
- ▶ Also,  $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$



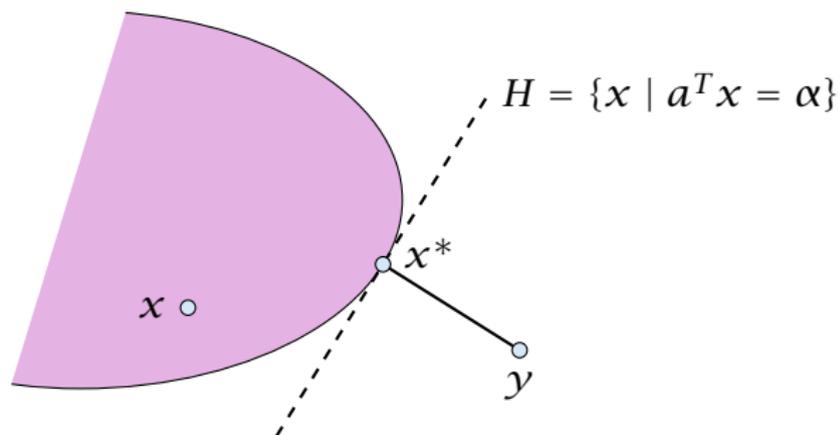
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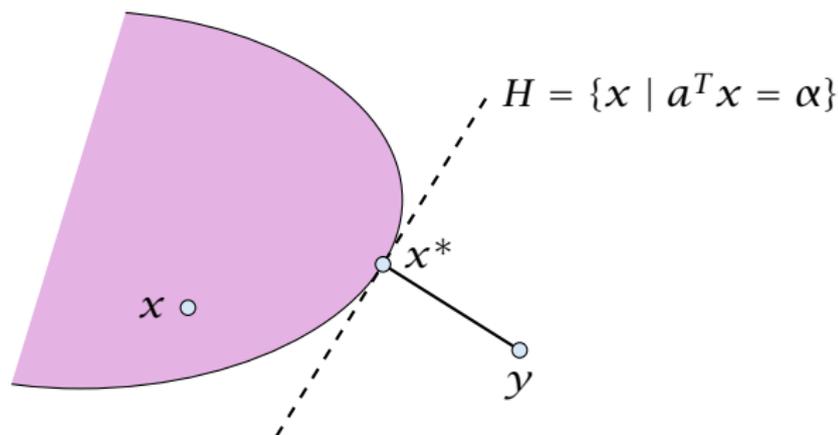
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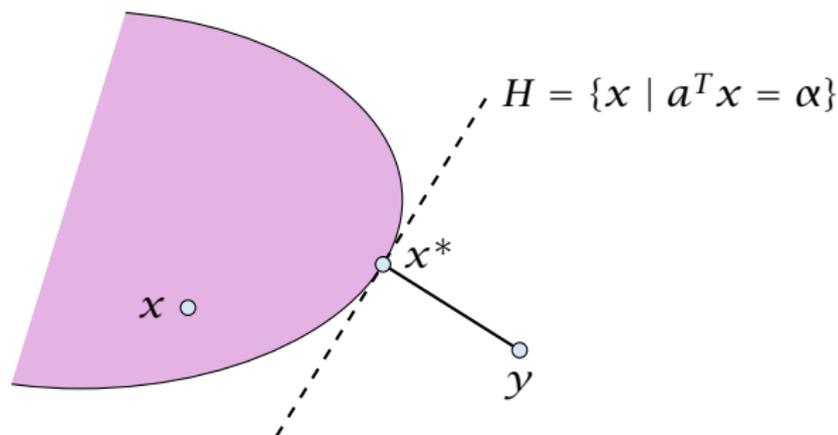
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## Lemma 9 (Farkas Lemma)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then *exactly one* of the following statements holds.

1.  $\exists x \in \mathbb{R}^n$  with  $Ax = b$ ,  $x \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $A^T y \geq 0$ ,  $b^T y < 0$

Assume  $\hat{x}$  satisfies 1. and  $\hat{y}$  satisfies 2. Then

$$0 > \hat{y}^T b = \hat{y}^T A \hat{x} \geq 0$$

Hence, at most one of the statements can hold.

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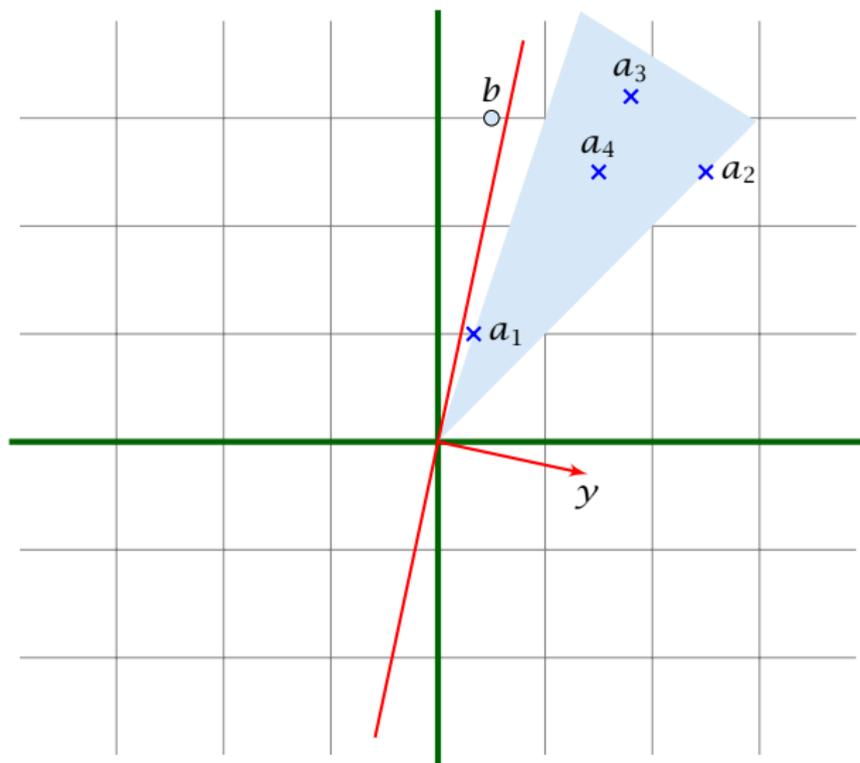
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## Farkas Lemma



If  $b$  is not in the cone generated by the columns of  $A$ , there exists a hyperplane  $y$  that separates  $b$  from the cone.

## Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider  $S = \{Ax : x \geq 0\}$  so that  $S$  closed, convex,  $b \notin S$ .

We want to show that there is  $y$  with  $A^T y \geq 0$ ,  $b^T y < 0$ .

Let  $y$  be a hyperplane that separates  $b$  from  $S$ . Hence,  $y^T b < \alpha$  and  $y^T s \geq \alpha$  for all  $s \in S$ .

$$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$$

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## Lemma 10 (Farkas Lemma; different version)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then exactly one of the following statements holds.

1.  $\exists x \in \mathbb{R}^n$  with  $Ax \leq b, x \geq 0$
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Rewrite the conditions:

1.  $\exists x \in \mathbb{R}^n$  with  $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b, x \geq 0, s \geq 0$
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# Proof of Strong Duality

$$P: z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

$$D: w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

## Theorem 11 (Strong Duality)

Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z$  and  $w$  denote the optimal solution to  $P$  and  $D$ , respectively (i.e.,  $P$  and  $D$  are non-empty). Then

$$z = w .$$

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$$\exists x \in \mathbb{R}^n$$

$$\text{s.t.} \quad Ax \leq b$$

$$-c^T x \leq -\alpha$$

$$x \geq 0$$

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$$\exists x \in \mathbb{R}^n$$

$$\begin{aligned} \text{s.t.} \quad Ax &\leq b \\ -c^T x &\leq -\alpha \\ x &\geq 0 \end{aligned}$$

$$\exists y \in \mathbb{R}^m; v \in \mathbb{R}$$

$$\begin{aligned} \text{s.t.} \quad A^T y - cv &\geq 0 \\ b^T y - \alpha v &< 0 \\ y, v &\geq 0 \end{aligned}$$

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From the definition of  $\alpha$  we know that the first system is infeasible; hence the second must be feasible.

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If the solution  $\mathbf{y}, \mathbf{v}$  has  $\mathbf{v} = 0$  we have that

$$\begin{aligned} \exists \mathbf{y} \in \mathbb{R}^m \\ \text{s.t. } A^T \mathbf{y} &\geq 0 \\ b^T \mathbf{y} &< 0 \\ \mathbf{y} &\geq 0 \end{aligned}$$

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is feasible. By Farkas lemma this gives that LP  $P$  is infeasible.  
Contradiction to the assumption of the lemma.

# Proof of Strong Duality

Hence, there exists a solution  $y, v$  with  $v > 0$ .

We can rescale this solution (scaling both  $y$  and  $v$ ) s.t.  $v = 1$ .

Then  $y$  is feasible for the dual but  $b^T y < \alpha$ . This means that  $w < \alpha$ .

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# Fundamental Questions

## Definition 12 (Linear Programming Problem (LP))

Let  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Q}^m$ ,  $c \in \mathbb{Q}^n$ ,  $\alpha \in \mathbb{Q}$ . Does there exist  $x \in \mathbb{Q}^n$  s.t.  $Ax = b$ ,  $x \geq 0$ ,  $c^T x \geq \alpha$ ?

### Questions:

- ▶ Is LP in NP?
- ▶ Is LP in co-NP? **yes!**
- ▶ Is LP in P?

Proof:

# Fundamental Questions

## Definition 12 (Linear Programming Problem (LP))

Let  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Q}^m$ ,  $c \in \mathbb{Q}^n$ ,  $\alpha \in \mathbb{Q}$ . Does there exist  $x \in \mathbb{Q}^n$  s.t.  $Ax = b$ ,  $x \geq 0$ ,  $c^T x \geq \alpha$ ?

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  - ▶ A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost  $< \alpha$ .

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# Complementary Slackness

## Lemma 13

Assume a linear program  $P = \max\{c^T x \mid Ax \leq b; x \geq 0\}$  has solution  $x^*$  and its dual  $D = \min\{b^T y \mid A^T y \geq c; y \geq 0\}$  has solution  $y^*$ .

1. If  $x_j^* > 0$  then the  $j$ -th constraint in  $D$  is tight.
2. If the  $j$ -th constraint in  $D$  is not tight then  $x_j^* = 0$ .
3. If  $y_i^* > 0$  then the  $i$ -th constraint in  $P$  is tight.
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If we say that a variable  $x_j^*$  ( $y_i^*$ ) has slack if  $x_j^* > 0$  ( $y_i^* > 0$ ), (i.e., the corresponding variable restriction is not tight) and a constraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

## Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \leq y^{*T} A x^* \leq b^T y^*$$

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From the constraint of the dual it follows that  $y^T A \geq c^T$ . Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g.  $(y^T A - c^T)_j > 0$  (the  $j$ -th constraint in the dual is not tight) then  $x_j = 0$  (2.). The result for (1./3./4.) follows similarly.

# Interpretation of Dual Variables

- ▶ Brewer: find mix of ale and beer that maximizes profits

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

- ▶ Entrepreneur: buy resources from brewer at minimum cost  
 $C, H, M$ : unit price for corn, hops and malt.

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Note that brewer won't sell (at least not all) if e.g.  
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## Marginal Price:

- ▶ How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- ▶ We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by  $\epsilon_C$ ,  $\epsilon_H$ , and  $\epsilon_M$ , respectively.

The profit increases to  $\max\{c^T x \mid Ax \leq b + \epsilon; x \geq 0\}$ . Because of strong duality this is equal to

$$\begin{array}{ll} \min & (b^T + \epsilon^T)y \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array}$$

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If  $\epsilon$  is “small” enough then the optimum dual solution  $y^*$  might not change. Therefore the profit increases by  $\sum_i \epsilon_i y_i^*$ .

Therefore we can interpret the dual variables as **marginal prices**.

Note that with this interpretation, complementary slackness becomes obvious.

If the farmer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).

If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the farmer. Hence, it makes no sense to have a surplus of this resource. Therefore its slack must be zero.

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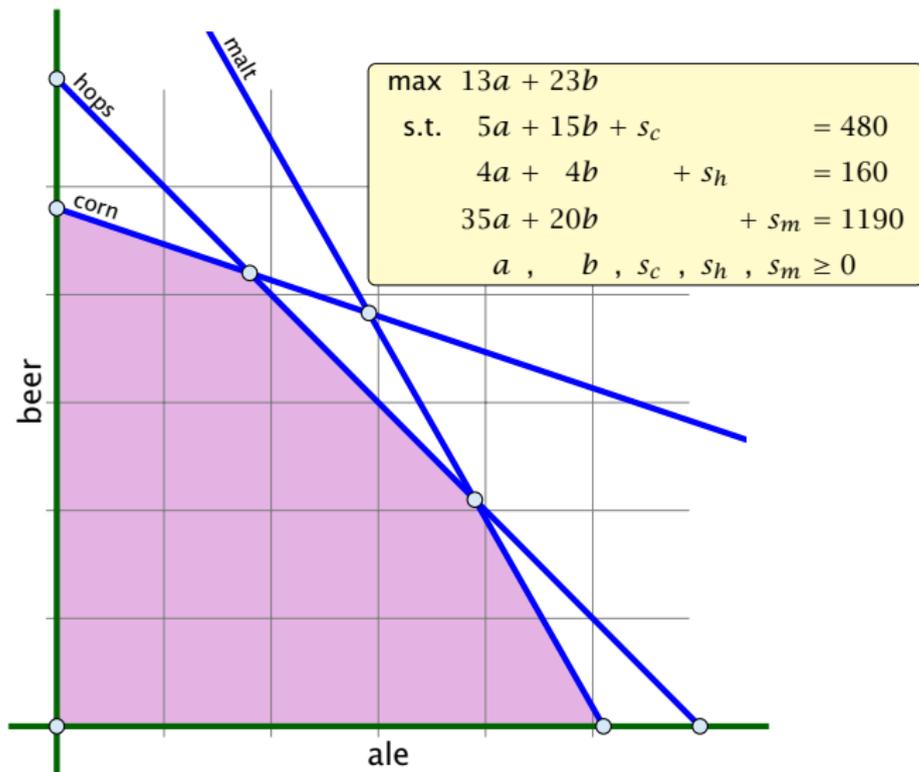
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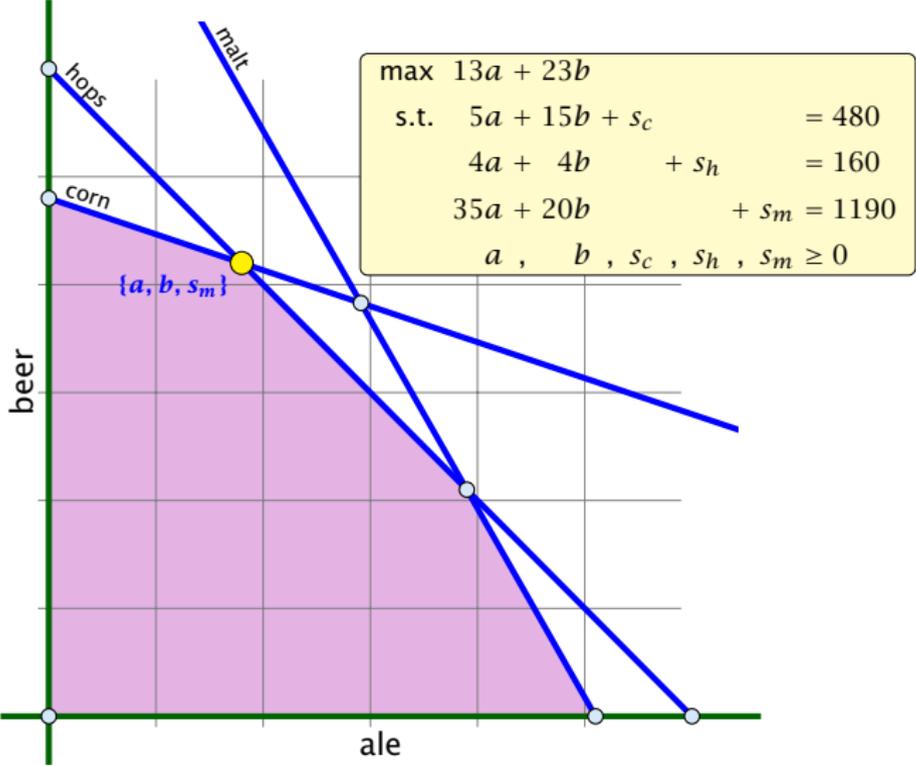
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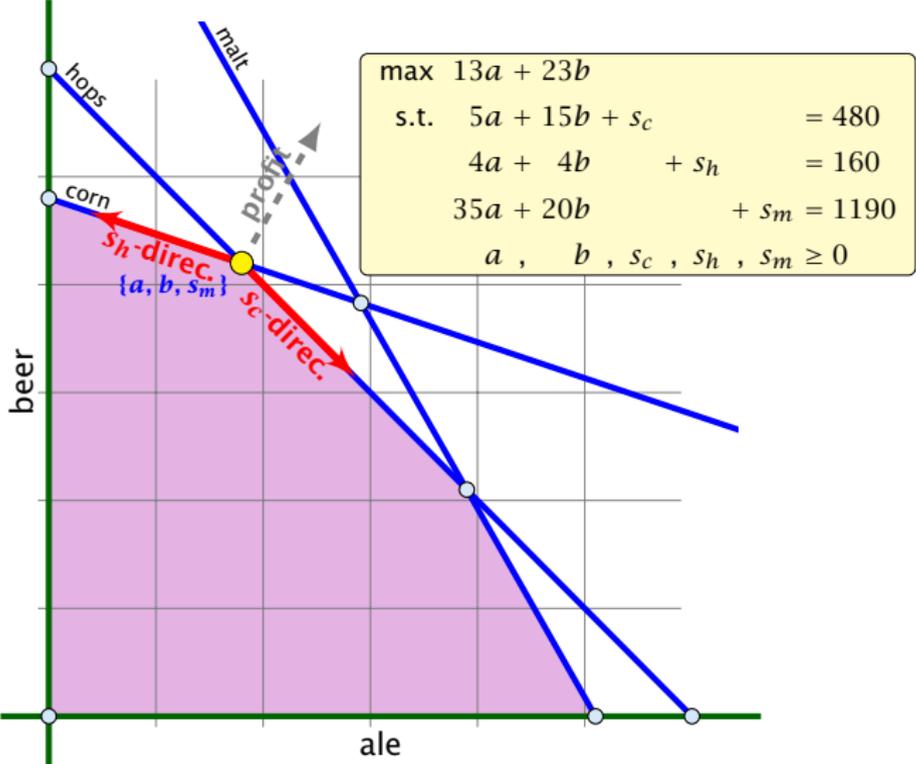
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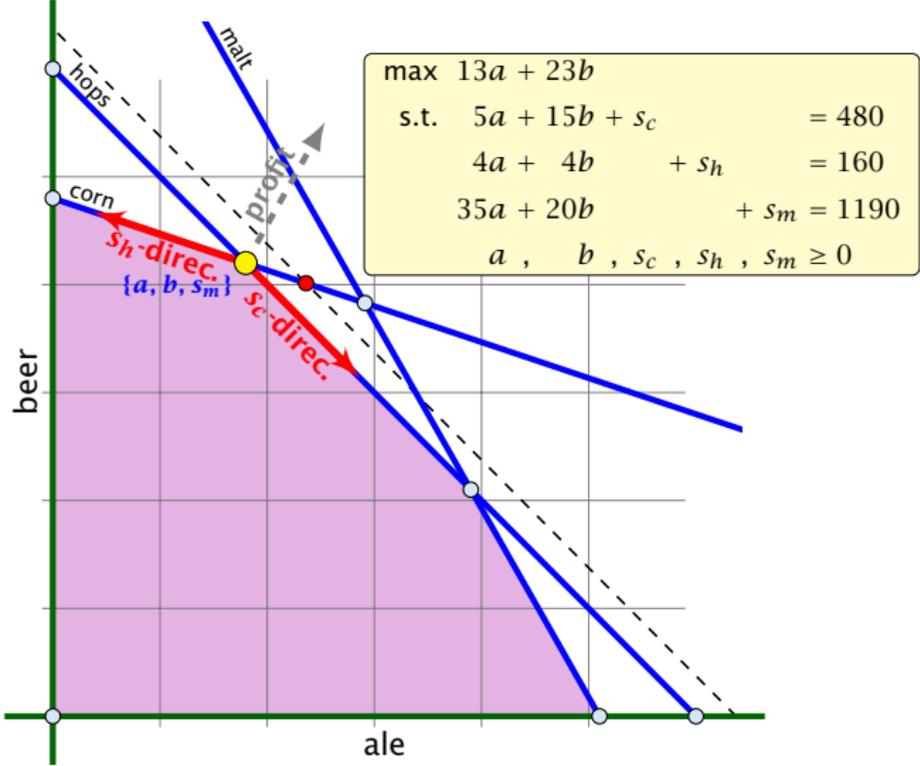
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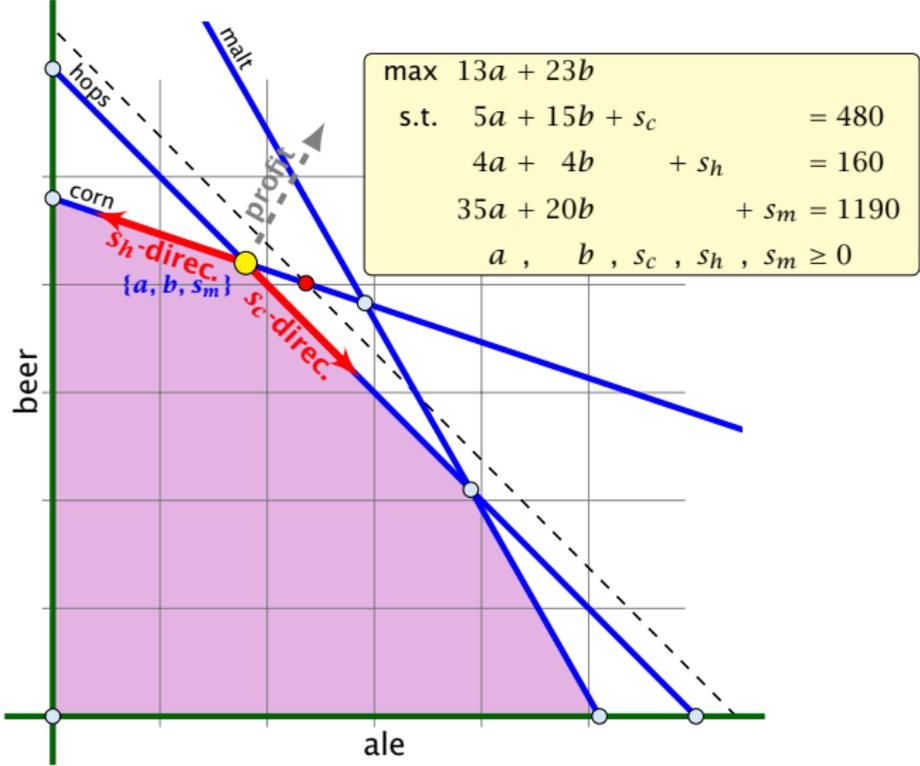
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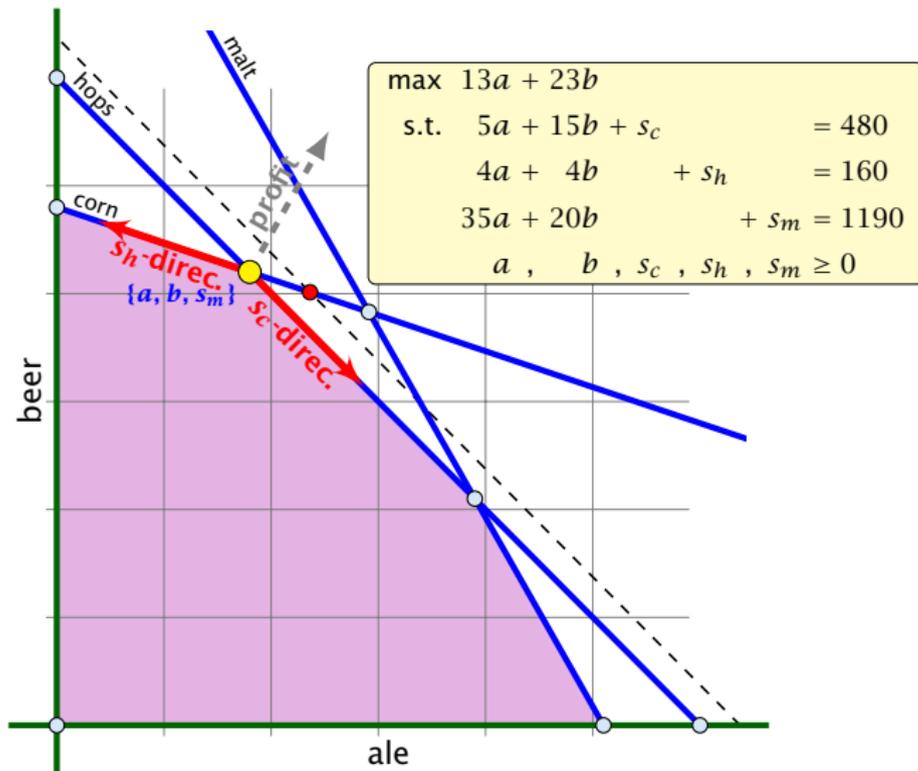
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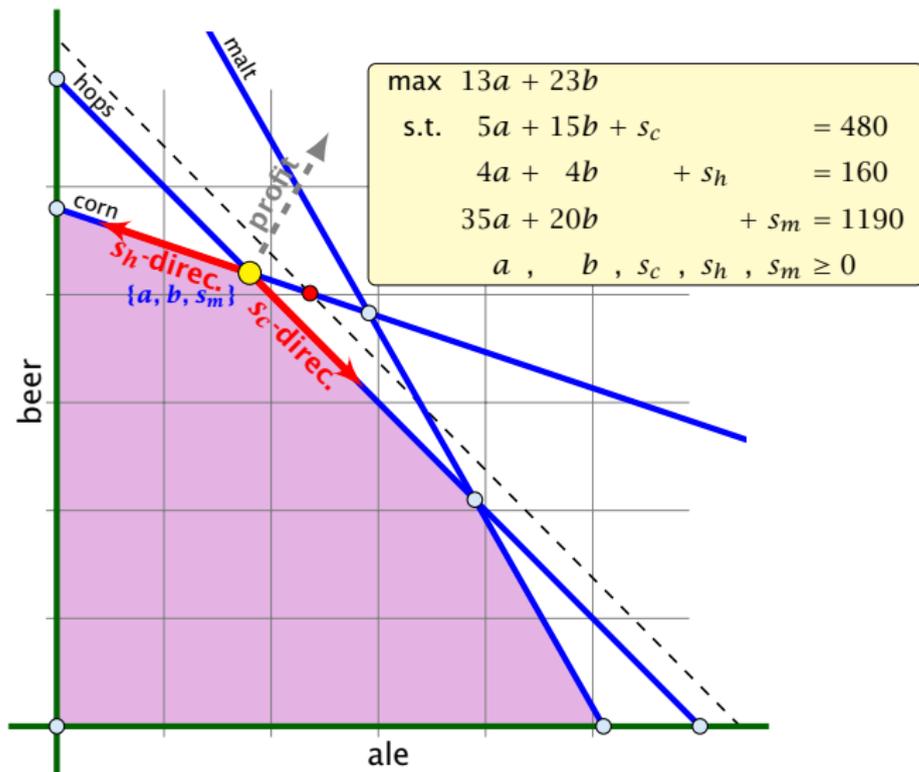
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The change in profit when increasing hops by one unit is

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Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

# Flows

## Definition 14

An  $(s, t)$ -flow in a (complete) directed graph  $G = (V, V \times V, c)$  is a function  $f : V \times V \rightarrow \mathbb{R}_0^+$  that satisfies

1. For each edge  $(x, y)$

$$0 \leq f_{xy} \leq c_{xy} .$$

(capacity constraints)

2. For each  $v \in V \setminus \{s, t\}$

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The **value of an  $(s, t)$ -flow  $f$**  is defined as

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Maximum Flow Problem:

Find an  $(s, t)$ -flow with maximum value.

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$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} \ (x, y \neq s, t): \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} \ (y \neq s, t): \quad 1\ell_{sy} \quad + 1p_y \geq 1 \\ & f_{xs} \ (x \neq s, t): \quad 1\ell_{xs} - 1p_x \quad \geq -1 \\ & f_{ty} \ (y \neq s, t): \quad 1\ell_{ty} \quad + 1p_y \geq 0 \\ & f_{xt} \ (x \neq s, t): \quad 1\ell_{xt} - 1p_x \quad \geq 0 \\ & f_{st}: \quad 1\ell_{st} \quad \geq 1 \\ & f_{ts}: \quad 1\ell_{ts} \quad \geq -1 \\ & \ell_{xy} \quad \geq 0 \end{array}$$

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We can interpret the  $\ell_{xy}$  value as assigning a length to every edge.

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This means  $p_x = 1$  or  $p_x = 0$  for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

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