

# Degeneracy Revisited

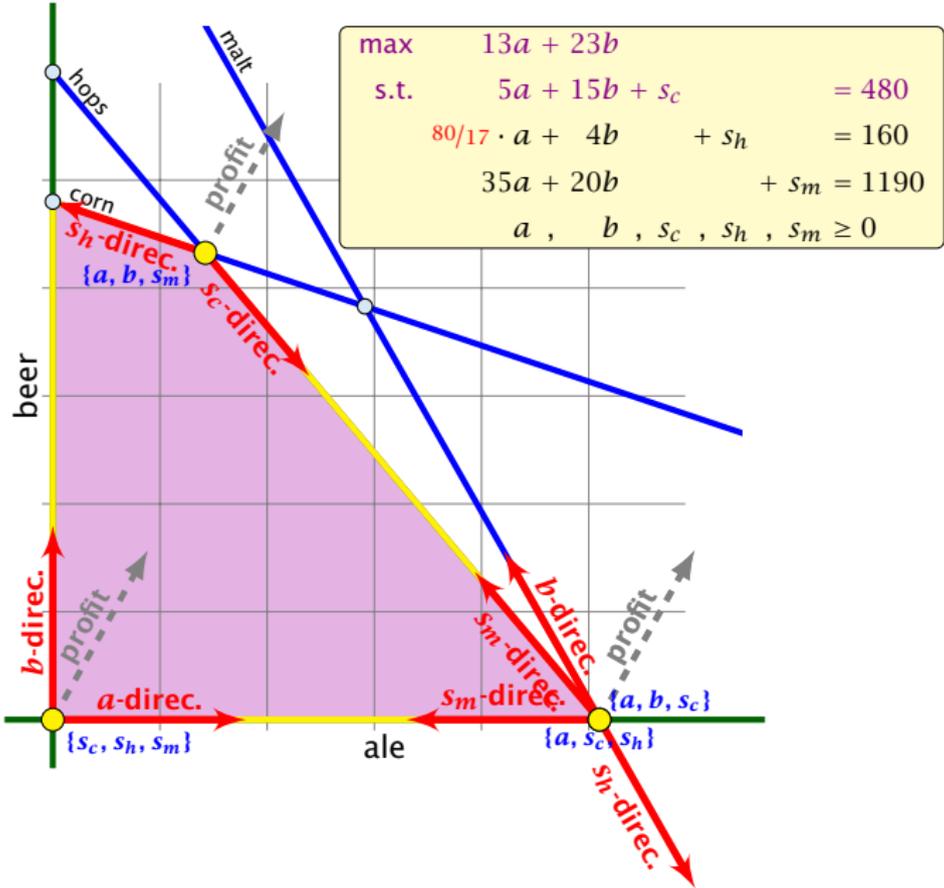
If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

## Idea:

Change  $LP := \max\{c^T x, Ax = b; x \geq 0\}$  into  $LP' := \max\{c^T x, Ax = b', x \geq 0\}$  such that

- I.  $LP$  is feasible
- II. If a set  $B$  of basis variables corresponds to an infeasible basis (i.e.  $A_B^{-1}b \not\geq 0$ ) then  $B$  corresponds to an infeasible basis in  $LP'$  (note that columns in  $A_B$  are linearly independent).
- III.  $LP$  has no degenerate basic solutions

# Degenerate Example



# Degeneracy Revisited

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

## Idea:

Given feasible  $LP := \max\{c^T x, Ax = b; x \geq 0\}$ . Change it into  $LP' := \max\{c^T x, Ax = b', x \geq 0\}$  such that

- I.  $LP'$  is feasible
- II. If a set  $B$  of basis variables corresponds to an infeasible basis (i.e.  $A_B^{-1}b \not\geq 0$ ) then  $B$  corresponds to an infeasible basis in  $LP'$  (note that columns in  $A_B$  are linearly independent).
- III.  $LP'$  has no degenerate basic solutions

# Perturbation

Let  $B$  be index set of **some** basis with basic solution

$$x_B^* = A_B^{-1}b \geq 0, x_N^* = 0 \quad (\text{i.e. } B \text{ is feasible})$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \quad \text{for } \varepsilon > 0 .$$

This is the perturbation that we are using.

# Property I

The new LP is feasible because the set  $B$  of basis variables provides a feasible basis:

$$A_B^{-1} \left( b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \right) = x_B^* + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \geq 0 .$$

## Property II

Let  $\tilde{B}$  be a non-feasible basis. This means  $(A_{\tilde{B}}^{-1}b)_i < 0$  for some row  $i$ .

Then for small enough  $\epsilon > 0$

$$\left( A_{\tilde{B}}^{-1} \left( b + A_B \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon^m \end{pmatrix} \right) \right)_i = (A_{\tilde{B}}^{-1}b)_i + \left( A_{\tilde{B}}^{-1}A_B \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon^m \end{pmatrix} \right)_i < 0$$

Hence,  $\tilde{B}$  is not feasible.

## Property III

Let  $\tilde{B}$  be a basis. It has an associated solution

$$x_{\tilde{B}}^* = A_{\tilde{B}}^{-1}b + A_{\tilde{B}}^{-1}A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynomial with variable  $\varepsilon$  of degree at most  $m$ .

$A_{\tilde{B}}^{-1}A_B$  has rank  $m$ . Therefore no polynomial is 0.

A polynomial of degree at most  $m$  has at most  $m$  roots (Nullstellen).

Hence,  $\varepsilon > 0$  small enough gives that no component of the above vector is 0. Hence, no degeneracies.

Since, there are no degeneracies Simplex will terminate when run on  $LP'$ .

- ▶ If it terminates because the reduced cost vector fulfills

$$\tilde{c} = (c^T - c_B^T A_B^{-1} A) \leq 0$$

then we have found an optimal basis. **Note that this basis is also optimal for  $LP$ , as the above constraint does not depend on  $b$ .**

- ▶ If it terminates because it finds a variable  $x_j$  with  $\tilde{c}_j > 0$  for which the  $j$ -th basis direction  $d$ , fulfills  $d \geq 0$  we know that  $LP'$  is unbounded. The basis direction **does not depend on  $b$** . Hence, we also know that  $LP$  is unbounded.

# Lexicographic Pivoting

Doing calculations with perturbed instances may be costly. Also the right choice of  $\varepsilon$  is difficult.

**Idea:**

Simulate behaviour of  $LP'$  without explicitly doing a perturbation.

# Lexicographic Pivoting

We choose the entering variable arbitrarily as before ( $\tilde{c}_e > 0$ , of course).

If we do not have a choice for the leaving variable then  $LP'$  and  $LP$  do the same (i.e., choose the same variable).

Otherwise we have to be careful.

# Lexicographic Pivoting

In the following we assume that  $b \geq 0$ . This can be obtained by replacing the initial system  $(A \mid b)$  by  $(A_B^{-1}A \mid A_B^{-1}b)$  where  $B$  is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

Then the perturbed instance is

$$b' = b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

## Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis  $B$  is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by  $x_N = 0, x_B = A_B^{-1} b$ .

If  $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$  we know that we have an optimum solution.

# Lexicographic Pivoting

LP chooses an arbitrary leaving variable that has  $\hat{A}_{\ell e} > 0$  and minimizes

$$\theta_{\ell} = \frac{\hat{b}_{\ell}}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_{\ell}}{(A_B^{-1}A_{*e})_{\ell}} .$$

$\ell$  is the index of a leaving variable within  $B$ . This means if e.g.  $B = \{1, 3, 7, 14\}$  and leaving variable is 3 then  $\ell = 2$ .

# Lexicographic Pivoting

## Definition 2

$u \leq_{\text{lex}} v$  if and only if the first component in which  $u$  and  $v$  differ fulfills  $u_i \leq v_i$ .

# Lexicographic Pivoting

LP' chooses an index that minimizes

$$\begin{aligned}\theta_\ell &= \frac{\left( A_B^{-1} \left( b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \right) \right)_\ell}{(A_B^{-1} A_{*e})_\ell} = \frac{\left( A_B^{-1}(b | I) \begin{pmatrix} 1 \\ \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \right)_\ell}{(A_B^{-1} A_{*e})_\ell} \\ &= \frac{\ell\text{-th row of } A_B^{-1}(b | I)}{(A_B^{-1} A_{*e})_\ell} \begin{pmatrix} 1 \\ \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}\end{aligned}$$

# Lexicographic Pivoting

This means you can choose the variable/row  $\ell$  for which the vector

$$\frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}$$

is lexicographically minimal.

Of course only including rows with  $(A_B^{-1}A_{*e})_\ell > 0$ .

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.