SS 2017

Efficient Algorithms and Data Structures II

Harald Räcke

Fakultät für Informatik
TU München

http://www14.in.tum.de/lehre/2017SS/ea/

Summer Term 2017

Part I

Organizational Matters

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- Modul: IN2004
- Name: “Efficient Algorithms and Data Structures II”
  “Effiziente Algorithmen und Datenstrukturen II”
- ECTS: 8 Credit points
- Lectures:
  - 4 SWS
    Wed 12:15–13:45 (Room 00.13.009A)
    Fri 10:15–11:45 (MS HS3)
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The Lecturer

- Harald Räcke
- Email: raecke@in.tum.de
- Room: 03.09.044
- Office hours: (per appointment)

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Part 2: Approximation Algorithms

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- R. Seidel:
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- D. Bertsimas and J.N. Tsitsiklis:
  Introduction to Linear Optimization,
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Part 1: Linear Programming

Part 2: Approximation Algorithms

- David P. Williamson and David B. Shmoys: *The Design of Approximation Algorithms*, Cambridge University Press 2011
- G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi: *Complexity and Approximation*, Springer, 1999

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Brewery Problem

Brewery brews ale and beer.

- Production limited by supply of corn, hops and barley malt
- Recipes for ale and beer require different amounts of resources

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
ale (barrel)	5	4	35	13
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- Introduce variables a and b that define how much ale and beer to produce.
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- 7.5 barrels ale, 29.5 barrels beer $\Longrightarrow$ 776 €
- 12 barrels ale, 28 barrels beer $\Longrightarrow$ 800 €

Standard Form LPs

LP in standard form:

Brewery Problem

Linear Program

- Introduce variables a and b that define how much ale and beer to produce.
- Choose the variables in such a way that the objective function (profit) is maximized.
- Make sure that no constraints (due to limited supply) are violated.

$$
\begin{array}{lrcrl}
\max & 13a & + & 23b & \\
\text{s.t.} & 5a & + & 15b & \leq 480 \\
 & 4a & + & 4b & \leq 160 \\
 & 35a & + & 20b & \leq 1190 \\
 & & & a, b & \geq 0
\end{array}
$$

Standard Form LPs

LP in standard form:

- input: numbers a_{ij}, c_j, b_i
- output: numbers x_j
- n = #decision variables, m = #constraints
- maximize linear objective function subject to linear (in)equalities

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Standard Form LPs

Original LP

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Standard Form

Add a slack variable to every constraint.

$$\begin{array}{lrcrcccccl} \max & 13a & + & 23b & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & = 480 \\ & 4a & + & 4b & & & + & s_h & & = 160 \\ & 35a & + & 20b & & & & & + s_m & = 1190 \\ & a & , & b & , & s_c & , & s_h & , s_m & \ge 0 \end{array}$$

Standard Form LPs

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Standard Form LPs

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Standard Form LPs

There are different standard forms:

standard form

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standard maximization form

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

standard minimization form

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

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Standard Form LPs

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Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{lrcl} \max & c^Tx & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \min & c^Tx & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

standard maximization form

$$\begin{array}{lrcl} \max & c^Tx & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

standard minimization form

$$\begin{array}{lrcl} \min & c^Tx & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **less or equal to equality:**

$$a - 3b + 5c \le 12 \implies \begin{array}{r} a - 3b + 5c + s = 12 \\ s \ge 0 \end{array}$$

- **greater or equal to equality:**

- **min to max:**

Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{lrcl} \max & c^Tx & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

$$\begin{array}{lrcl} \min & c^Tx & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

standard maximization form

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Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **less or equal to equality:**

$$a - 3b + 5c \le 12 \implies \begin{array}{rl} a - 3b + 5c + s &= 12 \\ s &\ge 0 \end{array}$$

- **greater or equal to equality:**

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Standard Form LPs

There are different standard forms:

standard form

$$\boxed{\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}}$$

$$\boxed{\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}}$$

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$$a - 3b + 5c \leq 12 \implies \begin{array}{r} a - 3b + 5c + s = 12 \\ s \geq 0 \end{array}$$

- **greater or equal to equality:**

$$a - 3b + 5c \geq 12 \implies \begin{array}{r} a - 3b + 5c - s = 12 \\ s \geq 0 \end{array}$$

- **min to max:**

Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{llcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

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- **min to max:**

Standard Form LPs

There are different standard forms:

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- **greater or equal to equality:**

$$a - 3b + 5c \ge 12 \;\Longrightarrow\; \begin{aligned} a - 3b + 5c - s &= 12 \\ s &\ge 0 \end{aligned}$$

- **min to max:**

$$\min a - 3b + 5c \;\Longrightarrow\; \max -a + 3b - 5c$$

Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

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Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **equality to less or equal:**

$$a - 3b + 5c = 12 \implies \begin{aligned} a - 3b + 5c &\leq 12 \\ -a + 3b - 5c &\leq -12 \end{aligned}$$

- **equality to greater or equal:**

- **unrestricted to nonnegative:**

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **less or equal to equality:**

$$a - 3b + 5c \leq 12 \implies \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\geq 0 \end{aligned}$$

- **greater or equal to equality:**

$$a - 3b + 5c \geq 12 \implies \begin{aligned} a - 3b + 5c - s &= 12 \\ s &\geq 0 \end{aligned}$$

- **min to max:**

$$\min a - 3b + 5c \implies \max -a + 3b - 5c$$

Standard Form LPs

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- **equality to greater or equal:**

- **unrestricted to nonnegative:**

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **less or equal to equality:**

$$a - 3b + 5c \le 12 \quad\Longrightarrow\quad \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\ge 0 \end{aligned}$$

- **greater or equal to equality:**

$$a - 3b + 5c \ge 12 \quad\Longrightarrow\quad \begin{aligned} a - 3b + 5c - s &= 12 \\ s &\ge 0 \end{aligned}$$

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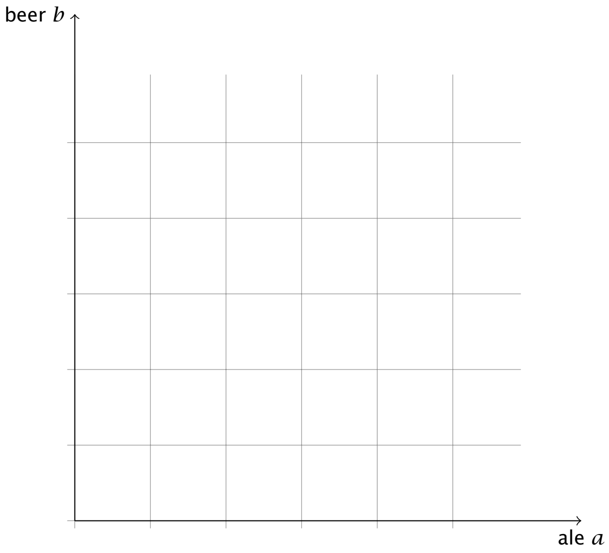

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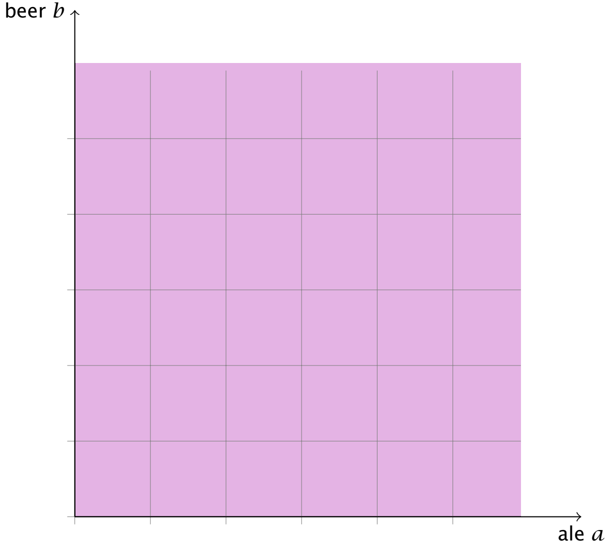

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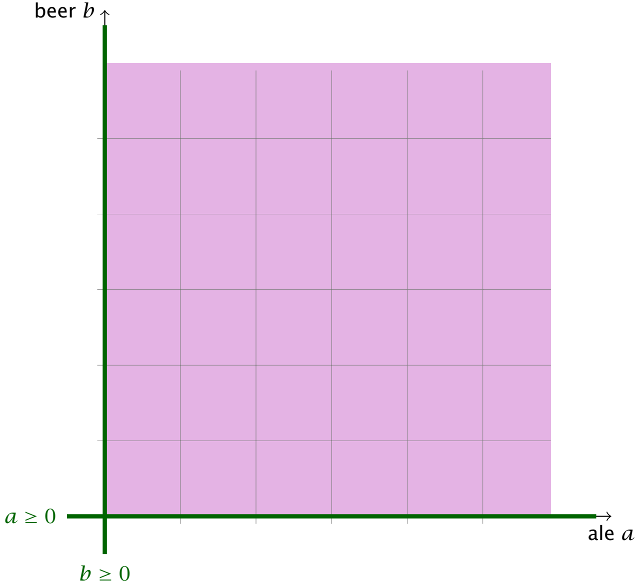

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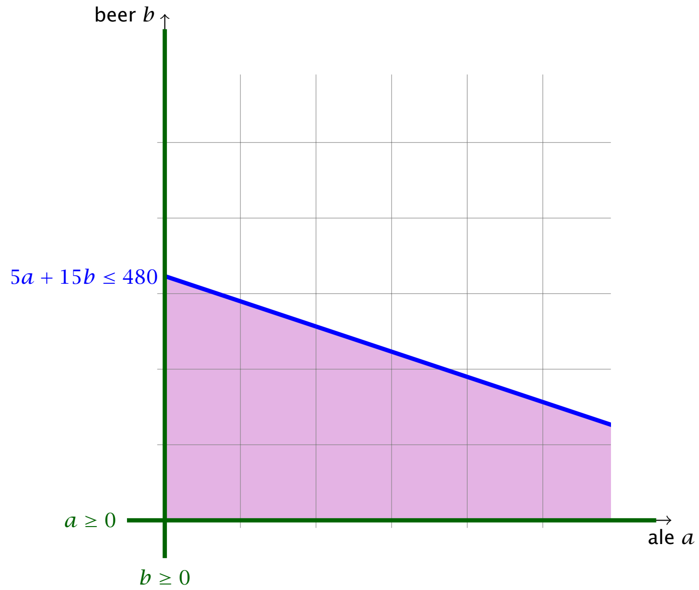

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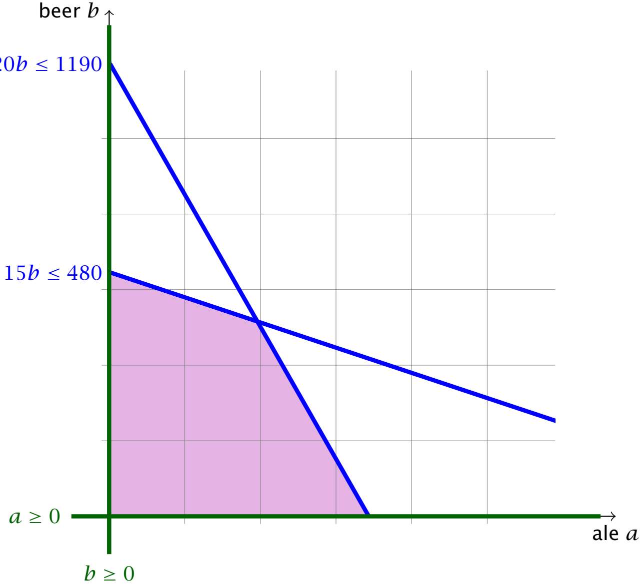

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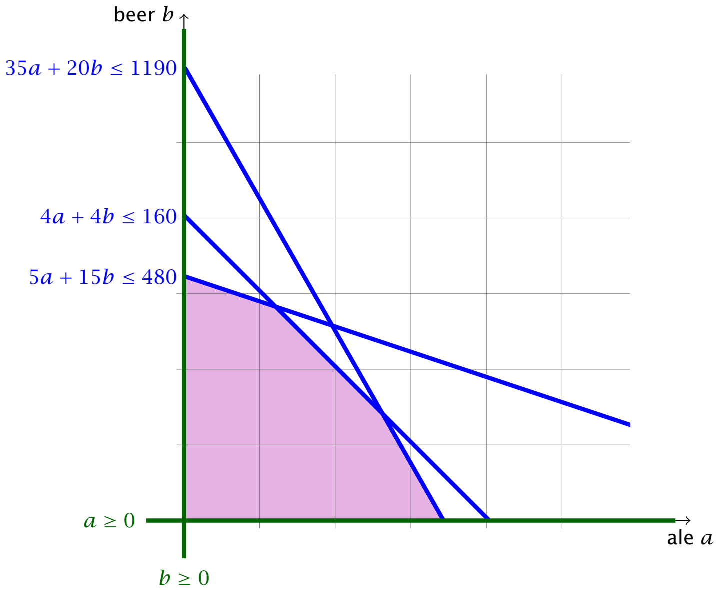

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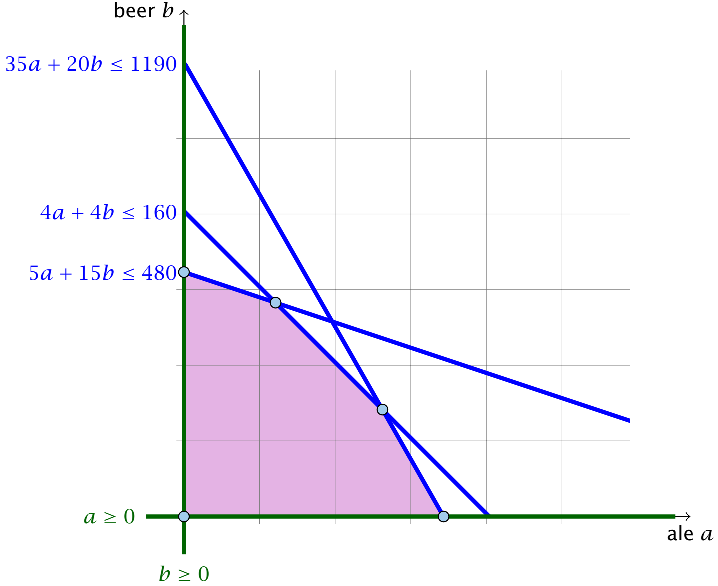

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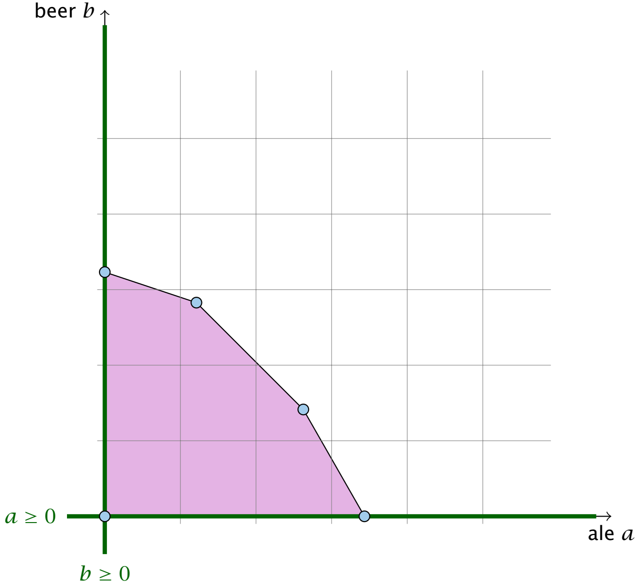

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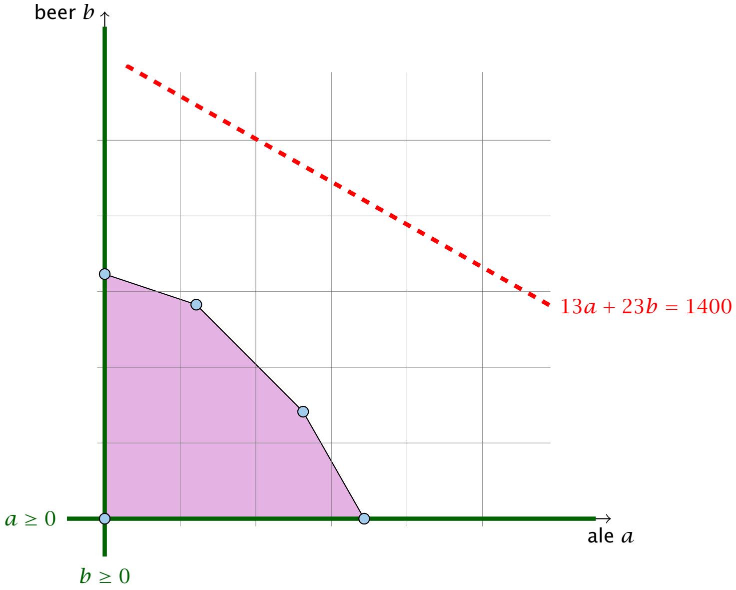

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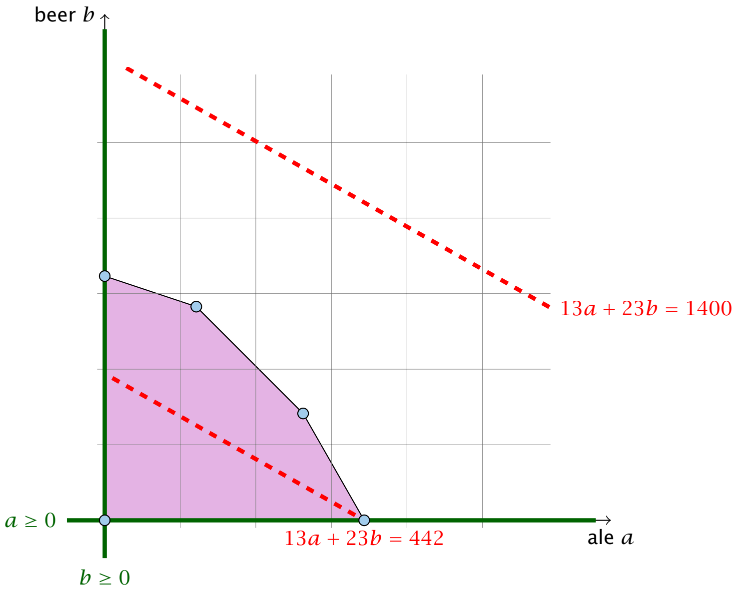

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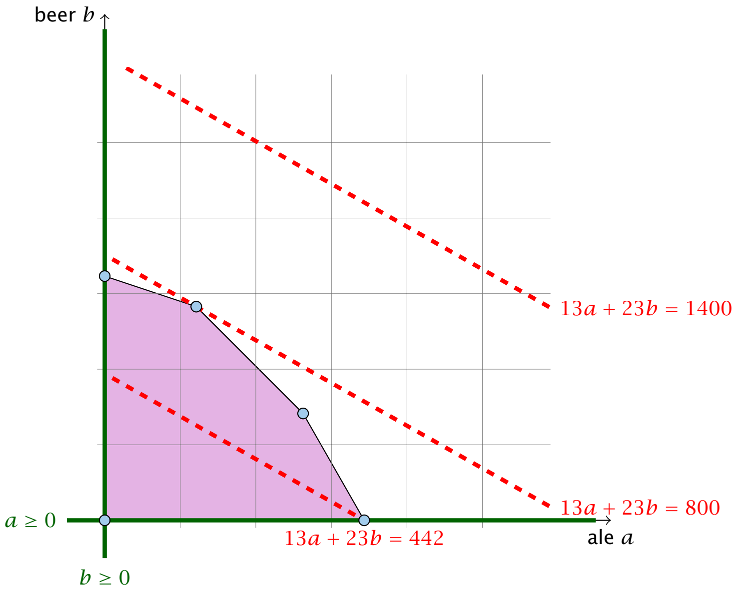

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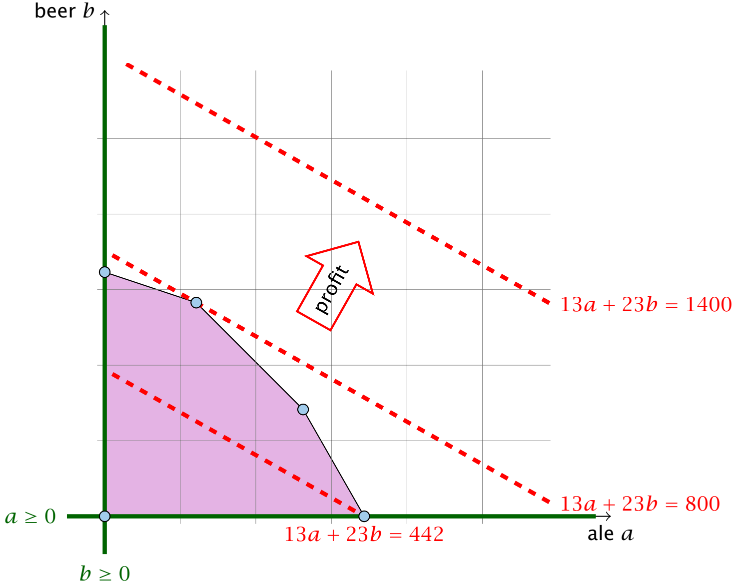

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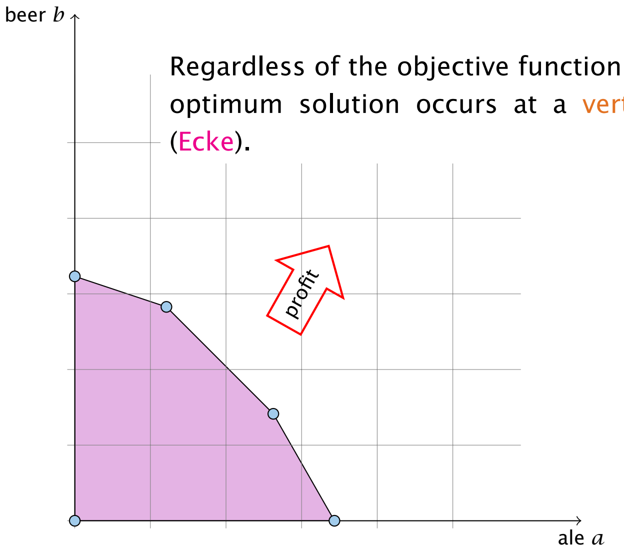

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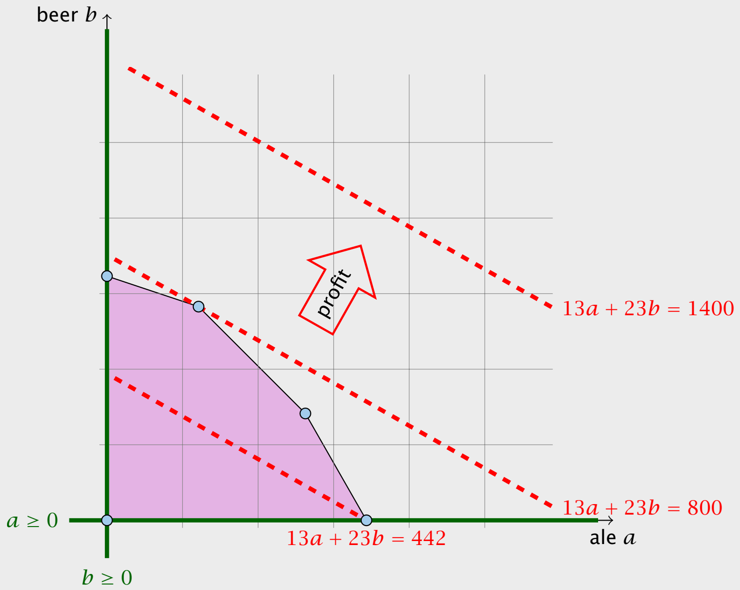

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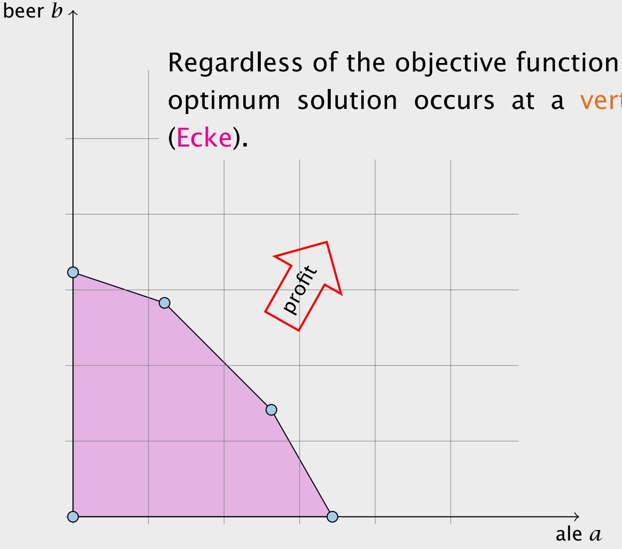

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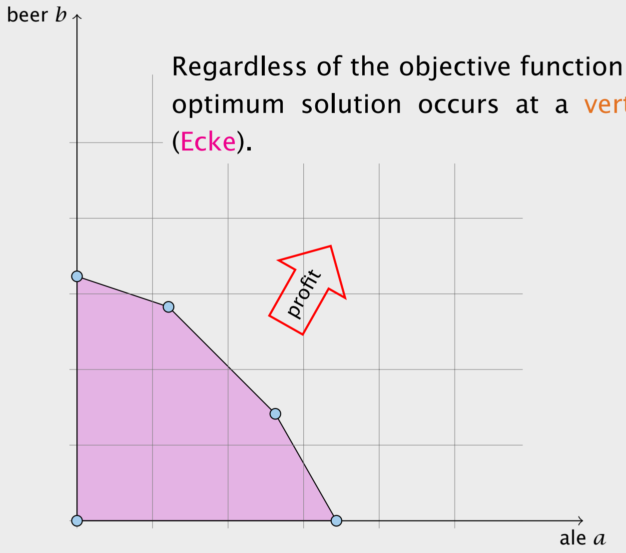

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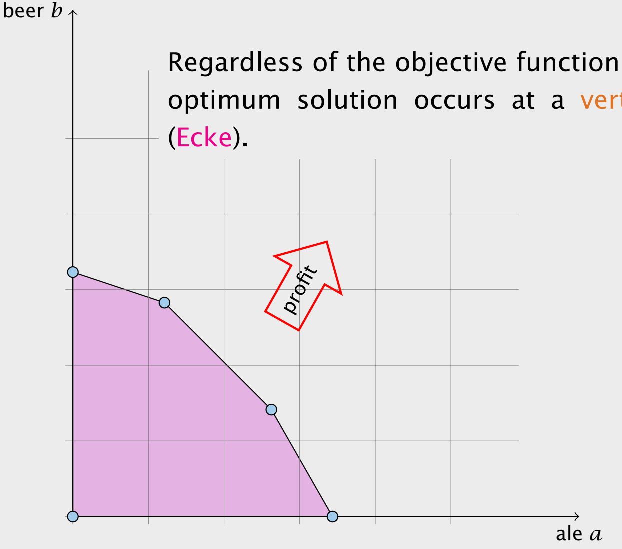

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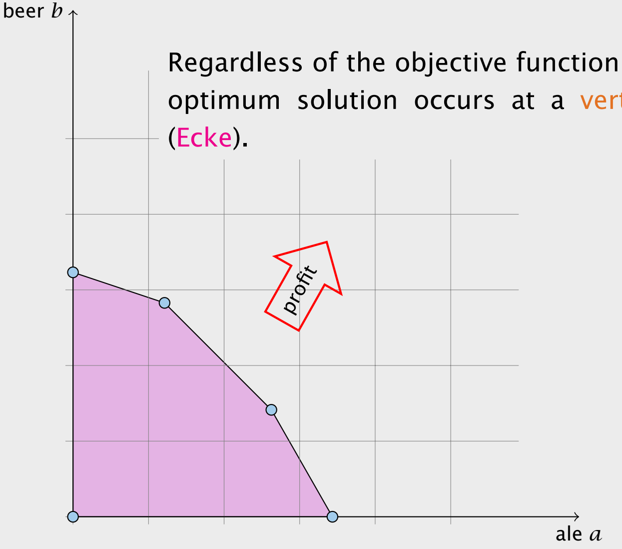

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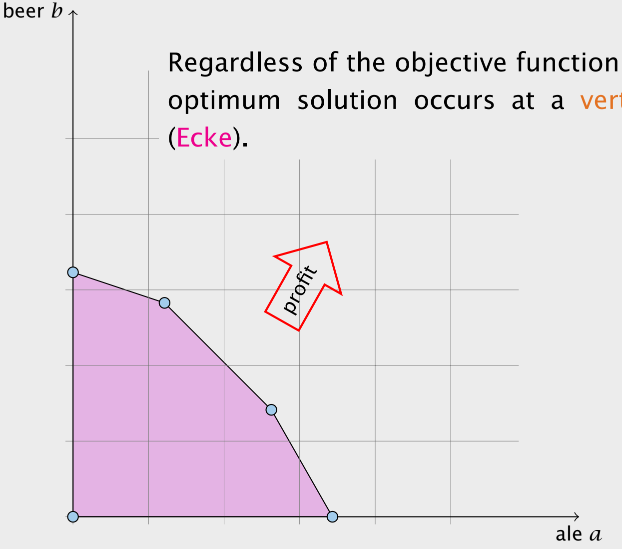

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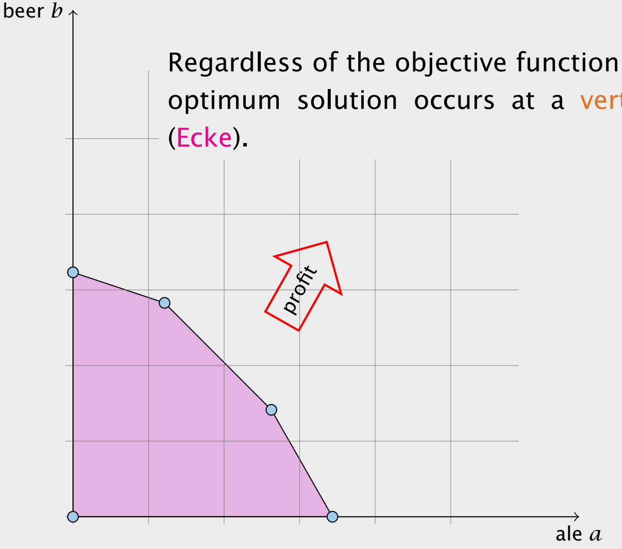

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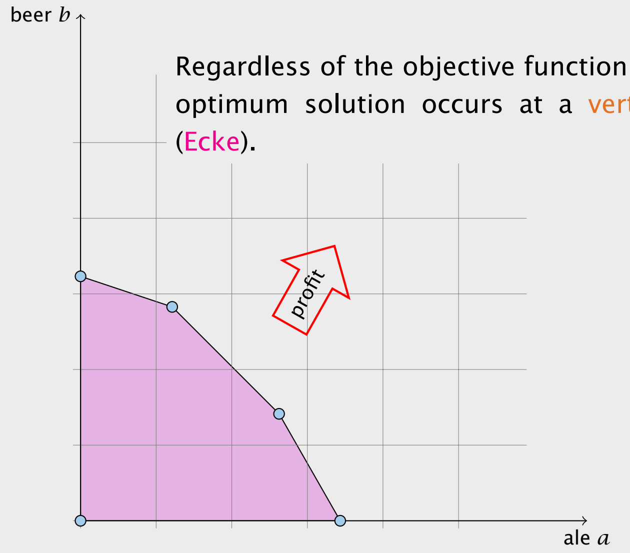

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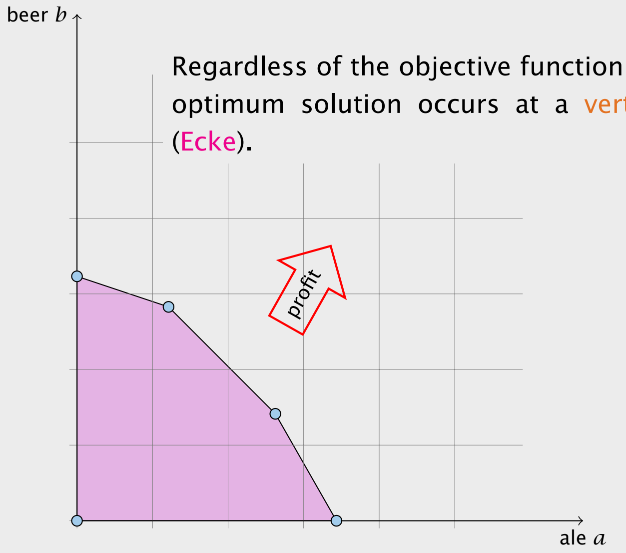

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Equivalent definition for vertex:

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Theorem 21
If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

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Proof

- suppose x is optimal solution that is not extreme point
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Convex Sets

Theorem 21
If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

- suppose x is optimal solution that is not extreme point
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Case 1. [$\exists j$ s.t. $d_j < 0$]

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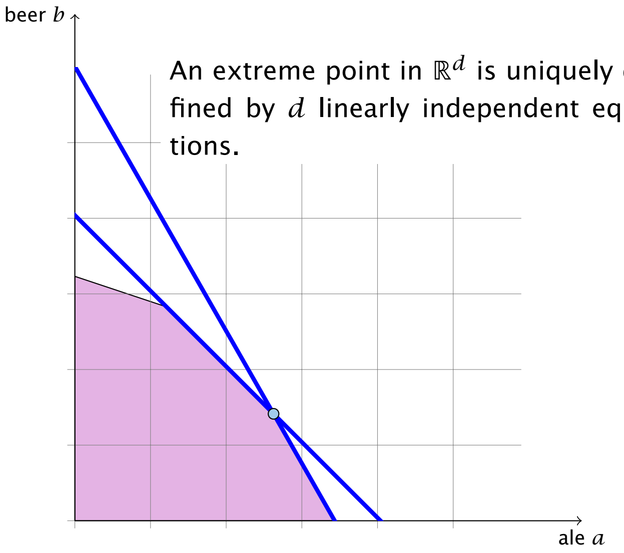
Algebraic View



An extreme point in $\mathbb{R}^d$ is uniquely defined by d linearly independent equations.

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Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

Theorem 22

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point iff A_B has linearly independent columns.

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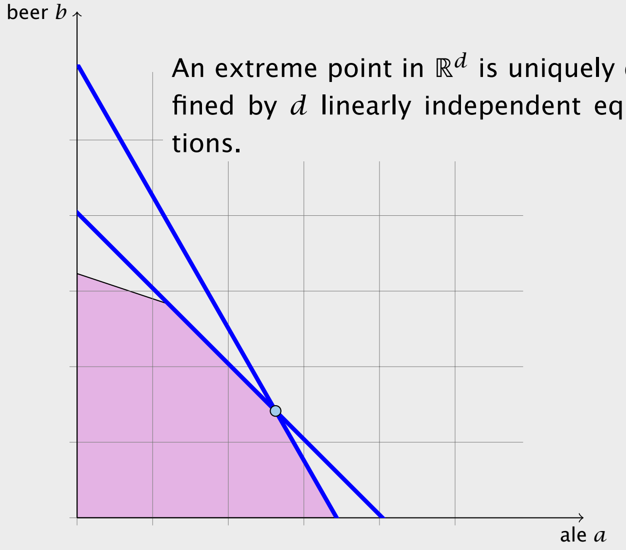

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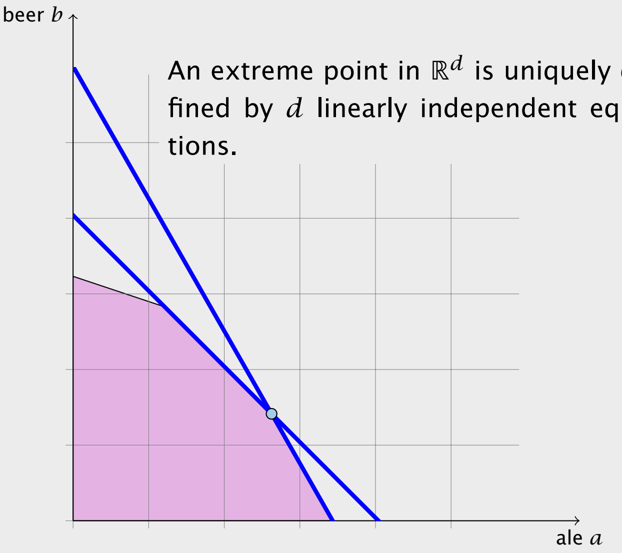

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- define $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
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- assume $c^T y = 0$; then $y_j = 0$ for all $j \notin B$
- $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - y_B) = 0$;
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Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P.

- define $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
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Theorem 22

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.*

Proof ($\Rightarrow$)

- assume A_B has linearly dependent columns
- there exists $d \neq 0$ such that $A_B d = 0$
- extend d to $\mathbb{R}^n$ by adding 0-components
- now, $Ad = 0$ and $d_j = 0$ whenever $x_j = 0$
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Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \ldots, n\}$ with $|B| = m$ and

- A_B is non-singular
- $x_B = A_B^{-1} b \geq 0$
- $x_N = 0$

where $N = \{1, \ldots, n\} \setminus B$.

Proof
Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\mathrm{rank}(A) = m$.

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$x \in \mathbb{R}^n$ is called basic solution (Basislösung) if $Ax = b$ and $\text{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a basic feasible solution (gültige Basislösung) if in addition $x \geq 0$.

A basis (Basis) is an index set $B \subseteq \{1, \ldots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$.

$x \in \mathbb{R}^n$ with $A_B x_B = b$ and $x_j = 0$ for all $j \notin B$ is the basic solution associated to basis B (die zu B assoziierte Basislösung)

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Theorem 24

Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \dots, n\}$ with $|B| = m$ and

- A_B is non-singular
- $x_B = A_B^{-1} b \geq 0$
- $x_N = 0$

where $N = \{1, \dots, n\} \setminus B$.

Proof

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\operatorname{rank}(A) = m$.

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A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i's are zero. The corresponding non-negativity constraint is fulfilled with equality.

Fact:
In a BFS at least n constraints are fulfilled with equality.

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For a general LP ($\max\{c^Tx \mid Ax \le b\}$) with n variables a point x is a basic feasible solution if x is feasible and there exist n (linearly independent) constraints that are tight.

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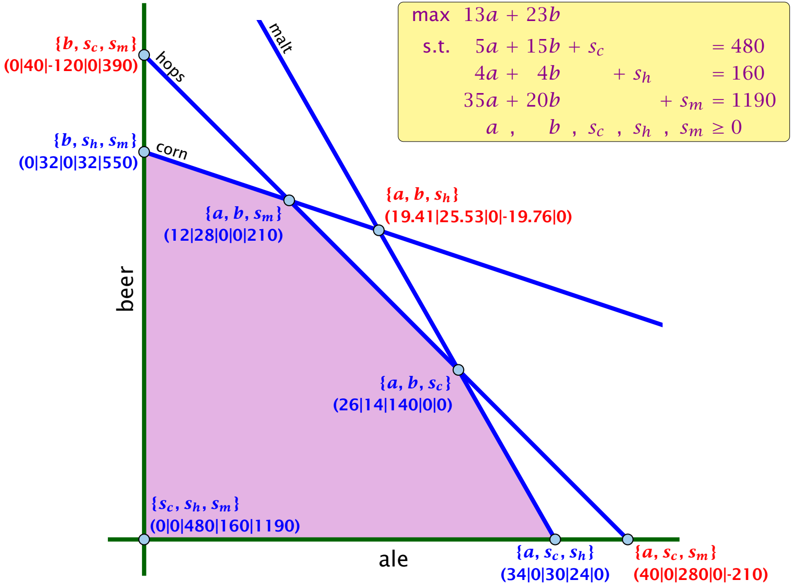

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Fundamental Questions

Linear Programming Problem (LP)

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^Tx \geq \alpha$?

Questions:

- Is LP in NP? yes!
- Is LP in co-NP?
- Is LP in P?

Proof:

- Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

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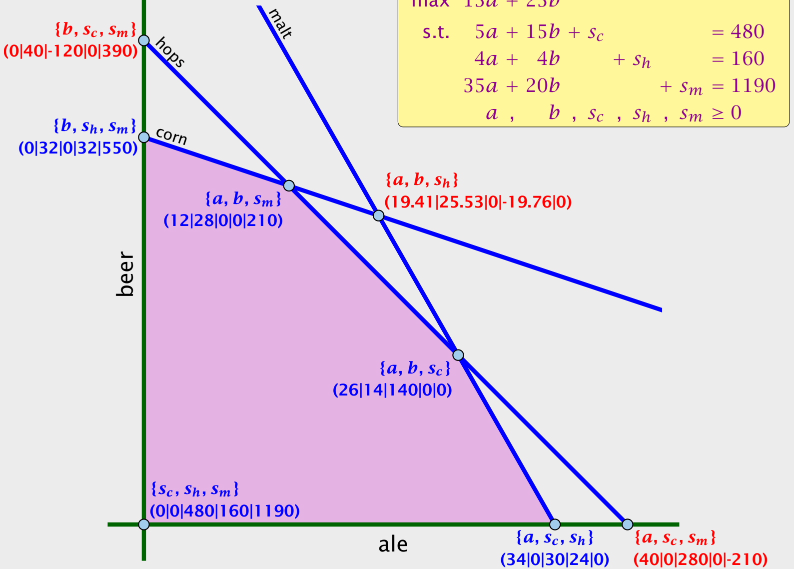

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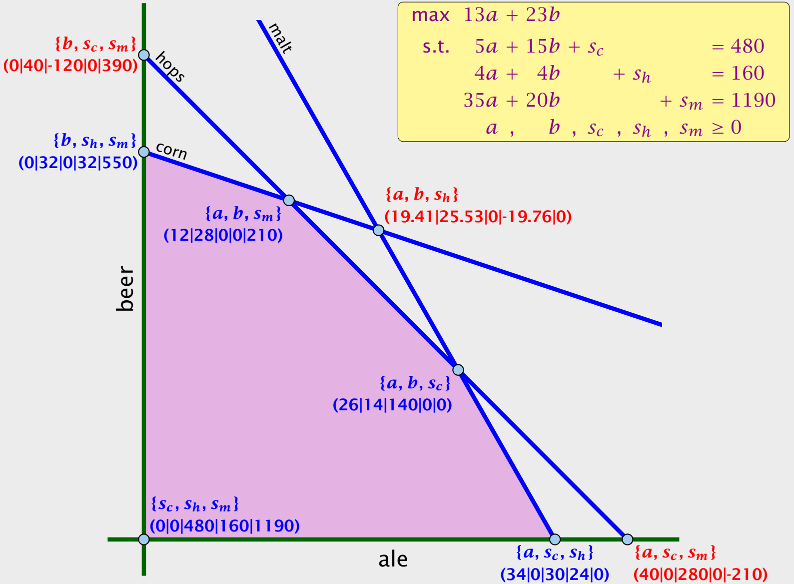

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Observation

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \text{poly}(n,m)\right)$.

- there are only $\binom{n}{m}$ different bases.
- compute the profit of each of them and take the maximum

What happens if LP is unbounded?

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4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

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\end{aligned}
$$

basis = $\{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$
\begin{array}{l}
\max\ Z \\
\begin{array}{rcl}
13a + 23\boldsymbol{b} & - Z & = 0 \\
5a + 15\boldsymbol{b} + s_c & & = 480 \\
4a + 4\boldsymbol{b} \quad + s_h & & = 160 \\
35a + 20\boldsymbol{b} \quad\quad + s_m & & = 1190 \\
a\ ,\ \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m & & \geq 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- Choose variable with coefficient > 0 as entering variable.

Pivoting Step

$$
\begin{array}{l}
\max\ Z \\
\begin{array}{rcl}
13a + 23b & - Z & = 0 \\
5a + 15b + s_c & & = 480 \\
4a + 4b \quad + s_h & & = 160 \\
35a + 20b \quad\quad + s_m & & = 1190 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & & \geq 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$
\begin{array}{l}
\max\ Z \\
\quad 13a + 23\boldsymbol{b} \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15\boldsymbol{b} + s_c \qquad\qquad = 480 \\
\quad 4a + 4\boldsymbol{b} \qquad + s_h \qquad = 160 \\
\quad 35a + 20\boldsymbol{b} \qquad\qquad + s_m = 1190 \\
\quad a\ ,\ \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.

Pivoting Step

$$
\begin{array}{l}
\max\ Z \\
\quad 13a + 23b \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15b + s_c \qquad\qquad = 480 \\
\quad 4a + 4b \qquad + s_h \qquad = 160 \\
\quad 35a + 20b \qquad\qquad + s_m = 1190 \\
\quad a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

max Z

$$
\begin{array}{rcl}
13a + 23\boldsymbol{b} & - Z & = 0 \\
5a + 15\boldsymbol{b} + s_c & & = 480 \\
4a + 4\boldsymbol{b} \quad + s_h & & = 160 \\
35a + 20\boldsymbol{b} \quad\quad + s_m & & = 1190 \\
a \;,\; \boldsymbol{b} \;,\; s_c \;,\; s_h \;,\; s_m & & \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.

Pivoting Step

max Z

$$
\begin{array}{rcl}
13a + 23b & - Z & = 0 \\
5a + 15b + s_c & & = 480 \\
4a + 4b \quad + s_h & & = 160 \\
35a + 20b \quad\quad + s_m & & = 1190 \\
a \;,\; b \;,\; s_c \;,\; s_h \;,\; s_m & & \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$\begin{array}{l}\max Z\\ \quad 13a + 23b \qquad\qquad\qquad\qquad - Z = 0\\ \quad 5a + 15\mathbf{b} + s_c \qquad\qquad\qquad = 480\\ \quad 4a + \ \ 4b \qquad + s_h \qquad\qquad = 160\\ \quad 35a + 20b \qquad\qquad + s_m \qquad = 1190\\ \quad a\ ,\ \ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad\qquad \geq 0\end{array}$$

$$\begin{array}{l}\text{basis} = \{s_c, s_h, s_m\}\\ a = b = 0\\ Z = 0\\ s_c = 480\\ s_h = 160\\ s_m = 1190\end{array}$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.

Pivoting Step

$$\begin{array}{l}\max Z\\ \quad 13a + 23b \qquad\qquad\qquad\qquad - Z = 0\\ \quad 5a + 15b + s_c \qquad\qquad\qquad = 480\\ \quad 4a + \ \ 4b \qquad + s_h \qquad\qquad = 160\\ \quad 35a + 20b \qquad\qquad + s_m \qquad = 1190\\ \quad a\ ,\ \ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad\qquad \geq 0\end{array}$$

$$\begin{array}{l}\text{basis} = \{s_c, s_h, s_m\}\\ a = b = 0\\ Z = 0\\ s_c = 480\\ s_h = 160\\ s_m = 1190\end{array}$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$\begin{array}{llllll} \max Z & & & & & \\ 13a + 23b & & & & -Z & = 0 \\ 5a + 15b + s_c & & & & & = 480 \\ 4a + 4b & + s_h & & & & = 160 \\ 35a + 20b & & + s_m & & & = 1190 \\ a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & & & & & \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

Pivoting Step

$$\begin{array}{llllll} \max Z & & & & & \\ 13a + 23b & & & & -Z & = 0 \\ 5a + 15b + s_c & & & & & = 480 \\ 4a + 4b & + s_h & & & & = 160 \\ 35a + 20b & & + s_m & & & = 1190 \\ a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & & & & & \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15b + s_c \qquad\qquad\qquad = 480 \\
\quad 4a + 4b \qquad + s_h \qquad\qquad = 160 \\
\quad 35a + 20b \qquad\qquad + s_m \qquad = 1190 \\
\quad a, \; b, \; s_c, \; s_h, \; s_m \qquad \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15\mathbf{b} + \mathbf{s_c} \qquad\qquad\qquad = 480 \\
\quad 4a + 4b \qquad + s_h \qquad\qquad = 160 \\
\quad 35a + 20b \qquad\qquad + s_m \qquad = 1190 \\
\quad a, \; b, \; s_c, \; s_h, \; s_m \qquad \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcl} 13a & + & 23\mathbf{b} & & & & & & & -Z & = 0 \\ 5a & + & 15\mathbf{b} & + & s_c & & & & & & = 480 \\ 4a & + & 4\mathbf{b} & & & + & s_h & & & & = 160 \\ 35a & + & 20\mathbf{b} & & & & & + & s_m & & = 1190 \\ a & , & \mathbf{b} & , & s_c & , & s_h & , & s_m & & \geq 0 \end{array} \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcl} 13a & + & 23b & & & & & & & -Z & = 0 \\ 5a & + & 15\mathbf{b} & + & \mathbf{s_c} & & & & & & = 480 \\ 4a & + & 4b & & & + & s_h & & & & = 160 \\ 35a & + & 20b & & & & & + & s_m & & = 1190 \\ a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0 \end{array} \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcl}
13a & + & 23\mathbf{b} & & & & & & & -Z & = 0 \\
5a & + & 15\mathbf{b} & + & s_c & & & & & & = 480 \\
4a & + & 4\mathbf{b} & & & + & s_h & & & & = 160 \\
35a & + & 20\mathbf{b} & & & & & + & s_m & & = 1190 \\
a & , & \mathbf{b} & , & s_c & , & s_h & , & s_m & & \geq 0
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & -Z & = -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & = 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & = 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & = 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcl}
13a & + & 23b & & & & & & & -Z & = 0 \\
5a & + & 15\mathbf{b} & + & \mathbf{s_c} & & & & & & = 480 \\
4a & + & 4b & & & + & s_h & & & & = 160 \\
35a & + & 20b & & & & & + & s_m & & = 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

max Z

$$\begin{aligned}
\tfrac{16}{3}a \quad - \tfrac{23}{15}s_c \quad - Z &= -736\\
\tfrac{1}{3}a + b + \tfrac{1}{15}s_c &= 32\\
\tfrac{8}{3}a \quad - \tfrac{4}{15}s_c + s_h &= 32\\
\tfrac{85}{3}a \quad - \tfrac{4}{3}s_c \quad + s_m &= 550\\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m &\ge 0
\end{aligned}$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

max Z

$$\begin{aligned}
13a + 23\mathbf{b} \quad - Z &= 0\\
5a + 15\mathbf{b} + s_c &= 480\\
4a + 4\mathbf{b} \quad + s_h &= 160\\
35a + 20\mathbf{b} \quad + s_m &= 1190\\
a\ ,\ \mathbf{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m &\ge 0
\end{aligned}$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

max Z

$$\begin{aligned}
\tfrac{16}{3}a \quad - \tfrac{23}{15}s_c \quad - Z &= -736\\
\tfrac{1}{3}a + b + \tfrac{1}{15}s_c &= 32\\
\tfrac{8}{3}a \quad - \tfrac{4}{15}s_c + s_h &= 32\\
\tfrac{85}{3}a \quad - \tfrac{4}{3}s_c \quad + s_m &= 550\\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m &\ge 0
\end{aligned}$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}\mathbf{a} & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}\mathbf{a} & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\mathbf{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}\mathbf{a} & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
\mathbf{a} & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

Choose variable a to bring into basis.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23\mathbf{b} & & & & & & & - & Z & = & 0 \\
5a & + & 15\mathbf{b} & + & s_c & & & & & & & = & 480 \\
4a & + & 4\mathbf{b} & & & + & s_h & & & & & = & 160 \\
35a & + & 20\mathbf{b} & & & & & + & s_m & & & = & 1190 \\
a & , & \mathbf{b} & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

$$\begin{array}{l} \max Z \\ \quad \tfrac{16}{3}\boldsymbol{a} \quad - \tfrac{23}{15}s_c \qquad\qquad - Z = -736 \\ \quad \tfrac{1}{3}\boldsymbol{a} + b + \tfrac{1}{15}s_c \qquad\qquad = 32 \\ \quad \tfrac{8}{3}\boldsymbol{a} \quad - \tfrac{4}{15}s_c + s_h \qquad = 32 \\ \quad \tfrac{85}{3}\boldsymbol{a} \quad - \tfrac{4}{3}s_c \quad + s_m \qquad = 550 \\ \quad \boldsymbol{a}, b, s_c, s_h, s_m \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{b, s_h, s_m\} \\ a = s_c = 0 \\ Z = 736 \\ b = 32 \\ s_h = 32 \\ s_m = 550 \end{array}$$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

$$\begin{array}{l} \max Z \\ \quad 13a + 23\boldsymbol{b} \qquad\qquad - Z = 0 \\ \quad 5a + 15\boldsymbol{b} + s_c \qquad\qquad = 480 \\ \quad 4a + 4\boldsymbol{b} \quad + s_h \qquad = 160 \\ \quad 35a + 20\boldsymbol{b} \quad + s_m \qquad = 1190 \\ \quad a, \boldsymbol{b}, s_c, s_h, s_m \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l} \max Z \\ \quad \tfrac{16}{3}a \quad - \tfrac{23}{15}s_c \qquad\qquad - Z = -736 \\ \quad \tfrac{1}{3}a + b + \tfrac{1}{15}s_c \qquad\qquad = 32 \\ \quad \tfrac{8}{3}a \quad - \tfrac{4}{15}s_c + s_h \qquad = 32 \\ \quad \tfrac{85}{3}a \quad - \tfrac{4}{3}s_c \quad + s_m \qquad = 550 \\ \quad a, b, s_c, s_h, s_m \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{b, s_h, s_m\} \\ a = s_c = 0 \\ Z = 736 \\ b = 32 \\ s_h = 32 \\ s_m = 550 \end{array}$$

$$\begin{array}{l} \max Z \\ \begin{array}{rcl} \frac{16}{3}a \quad - \frac{23}{15}s_c \quad\quad - Z & = & -736 \\ \frac{1}{3}a + b + \frac{1}{15}s_c & = & 32 \\ \frac{8}{3}\mathbf{a} \quad - \frac{4}{15}s_c + s_h & = & 32 \\ \frac{85}{3}a \quad - \frac{4}{3}s_c \quad + s_m & = & 550 \\ a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0 \end{array} \end{array}$$

$$\begin{array}{l} \text{basis} = \{b, s_h, s_m\} \\ a = s_c = 0 \\ Z = 736 \\ b = 32 \\ s_h = 32 \\ s_m = 550 \end{array}$$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \max Z \\ \begin{array}{rcl} 13a + 23\mathbf{b} \quad\quad - Z & = & 0 \\ 5a + 15\mathbf{b} + s_c & = & 480 \\ 4a + 4\mathbf{b} \quad + s_h & = & 160 \\ 35a + 20\mathbf{b} \quad\quad + s_m & = & 1190 \\ a\ ,\ \mathbf{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0 \end{array} \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l} \max Z \\ \begin{array}{rcl} \frac{16}{3}a \quad - \frac{23}{15}s_c \quad\quad - Z & = & -736 \\ \frac{1}{3}a + b + \frac{1}{15}s_c & = & 32 \\ \frac{8}{3}a \quad - \frac{4}{15}s_c + s_h & = & 32 \\ \frac{85}{3}a \quad - \frac{4}{3}s_c \quad + s_m & = & 550 \\ a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0 \end{array} \end{array}$$

$$\begin{array}{l} \text{basis} = \{b, s_h, s_m\} \\ a = s_c = 0 \\ Z = 736 \\ b = 32 \\ s_h = 32 \\ s_m = 550 \end{array}$$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\mathbf{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$

$b = 28$
$a = 12$
$s_m = 210$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23\mathbf{b} & & & & & & & - & Z & = & 0 \\
5a & + & 15\mathbf{b} & + & s_c & & & & & & & = & 480 \\
4a & + & 4\mathbf{b} & & & + & s_h & & & & & = & 160 \\
35a & + & 20\mathbf{b} & & & & & + & s_m & & & = & 1190 \\
a & , & \mathbf{b} & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{lcl}
\text{basis} & = & \{b, s_h, s_m\} \\
a & = & s_c = 0 \\
Z & = & 736 \\
b & = & 32 \\
s_h & = & 32 \\
s_m & = & 550
\end{array}
$$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{lcl}
\text{basis} & = & \{a, b, s_m\} \\
s_c & = & s_h = 0 \\
Z & = & 800 \\
b & = & 28 \\
a & = & 12 \\
s_m & = & 210
\end{array}
$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcl}
\frac{16}{3}a - \frac{23}{15}s_c - Z & = & -736 \\
\frac{1}{3}a + b + \frac{1}{15}s_c & = & 32 \\
\frac{8}{3}a - \frac{4}{15}s_c + s_h & = & 32 \\
\frac{85}{3}a - \frac{4}{3}s_c + s_m & = & 550 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{lcl}
\text{basis} & = & \{b, s_h, s_m\} \\
a & = & s_c = 0 \\
Z & = & 736 \\
b & = & 32 \\
s_h & = & 32 \\
s_m & = & 550
\end{array}
$$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcl}
- s_c - 2s_h - Z & = & -800 \\
b + \frac{1}{10}s_c - \frac{1}{8}s_h & = & 28 \\
a - \frac{1}{10}s_c + \frac{3}{8}s_h & = & 12 \\
\frac{3}{2}s_c - \frac{85}{8}s_h + s_m & = & 210 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{lcl}
\text{basis} & = & \{a, b, s_m\} \\
s_c & = & s_h = 0 \\
Z & = & 800 \\
b & = & 28 \\
a & = & 12 \\
s_m & = & 210
\end{array}
$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$

$b = 28$
$a = 12$
$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$

$b = 28$
$a = 12$
$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
\qquad
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, 3\cdot32/8, 3\cdot550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
\qquad
\begin{array}{l}
\text{basis} = \{a, b, s_m\} \\
s_c = s_h = 0 \\
Z = 800 \\
b = 28 \\
a = 12 \\
s_m = 210
\end{array}
$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h,\ s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcl} \frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - Z & = -736 \\ \frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & = 32 \\ \frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & = 32 \\ \frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & = 550 \\ a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcl} & & & - & s_c & - & 2s_h & & & - Z & = -800 \\ & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & = 28 \\ a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & = 12 \\ & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & = 210 \\ a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$

$b = 28$
$a = 12$
$s_m = 210$

Matrix View

Let our linear program be

$$
\begin{array}{rcrcl}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

$$
\begin{array}{rcrcl}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h,\ s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{cccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

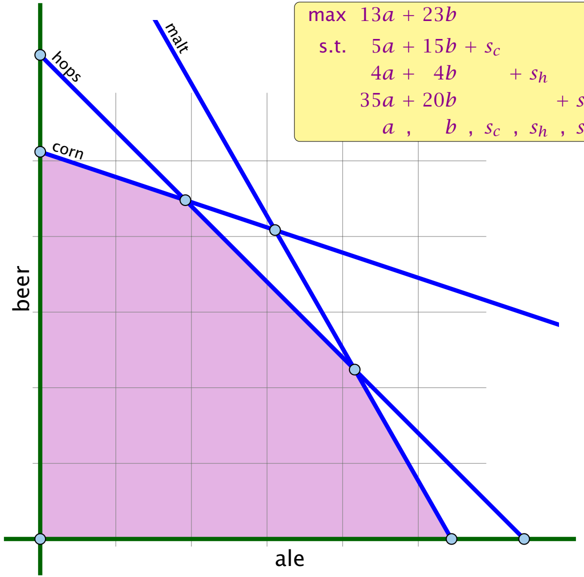
4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h,\ s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Geometric View of Pivoting



$$
\begin{array}{llllllll}
\max & 13a + 23b & & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & = & 480 \\
 & 4a + 4b & & + s_h & & = & 160 \\
 & 35a + 20b & & & + s_m & = & 1190 \\
 & a, \; b, & s_c, & s_h, & s_m & \geq & 0
\end{array}
$$

Matrix View

Let our linear program be

$$
\begin{array}{ccccc}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

$$
\begin{array}{ccccc}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

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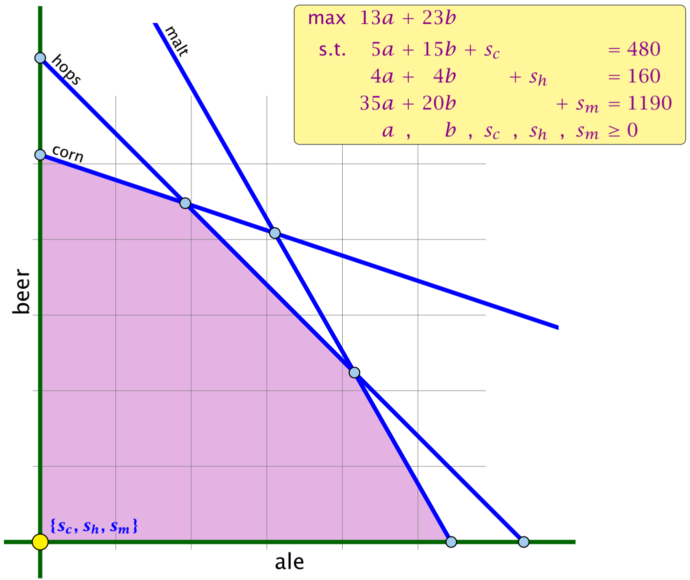
Geometric View of Pivoting



Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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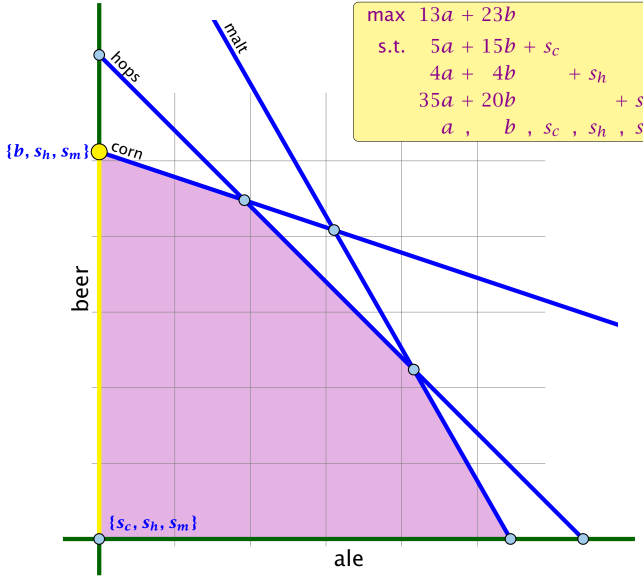

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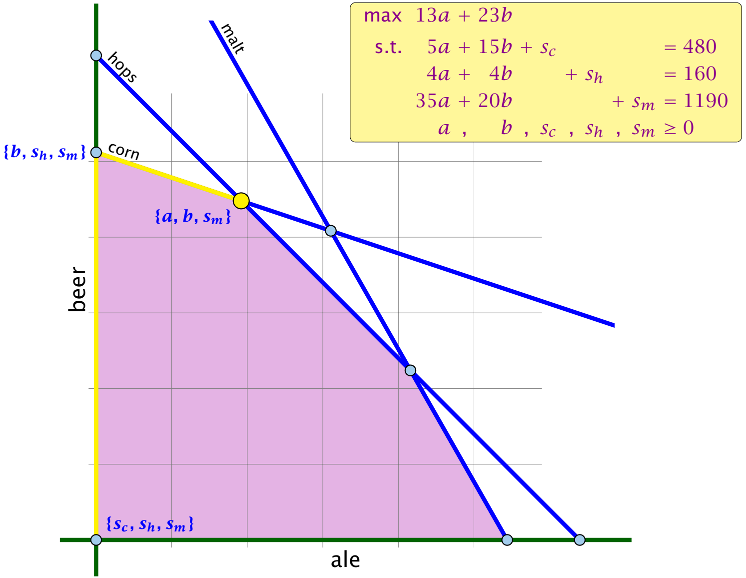

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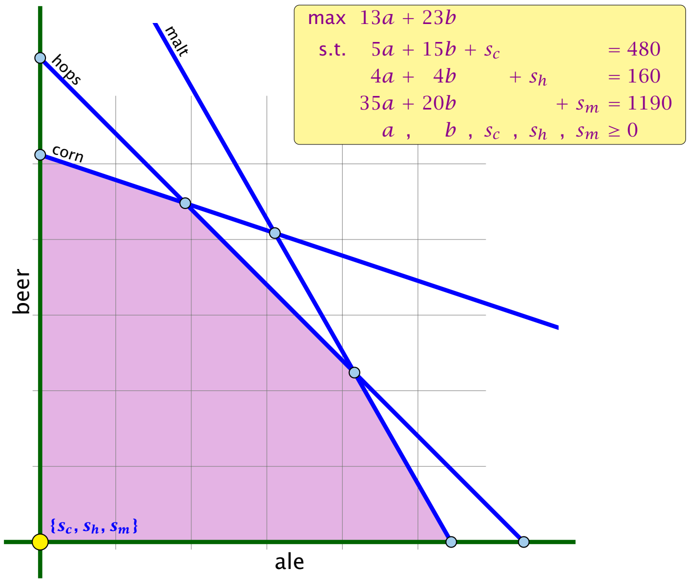

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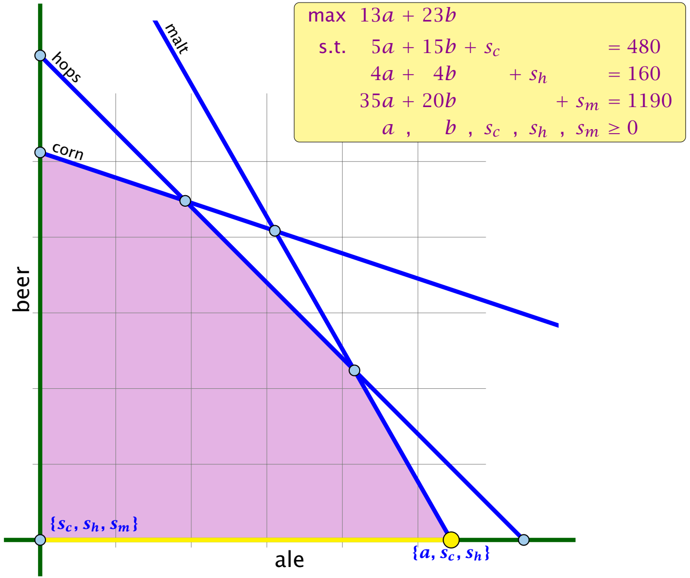

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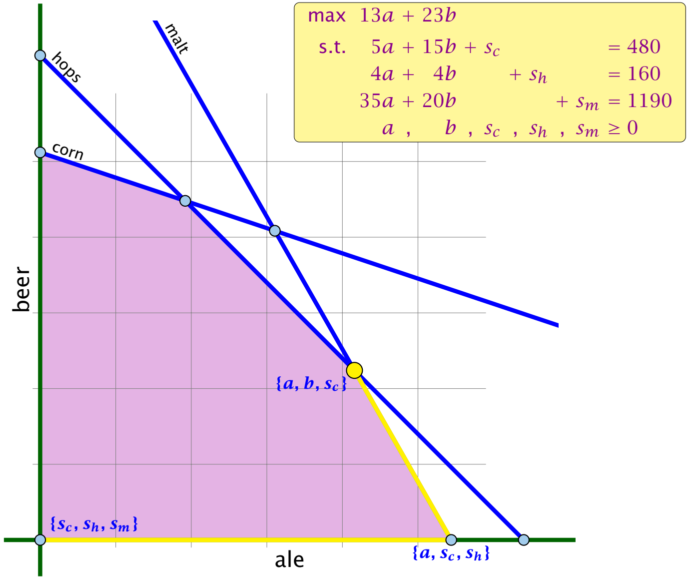

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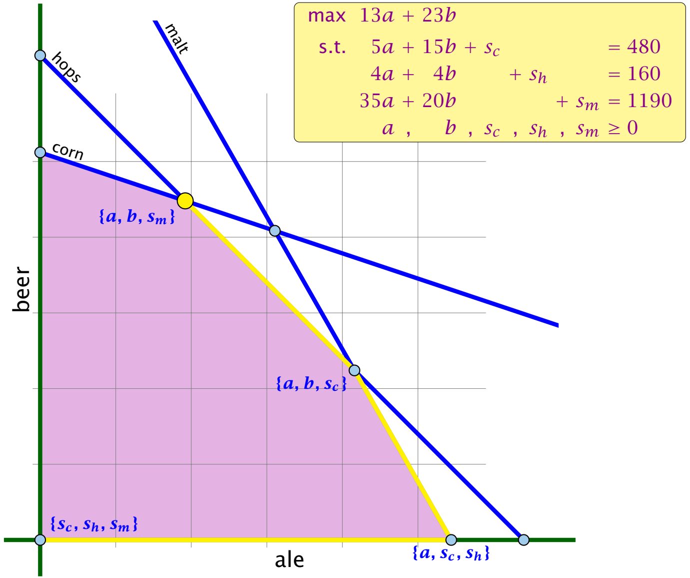

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- Given basis B with BFS x^*.
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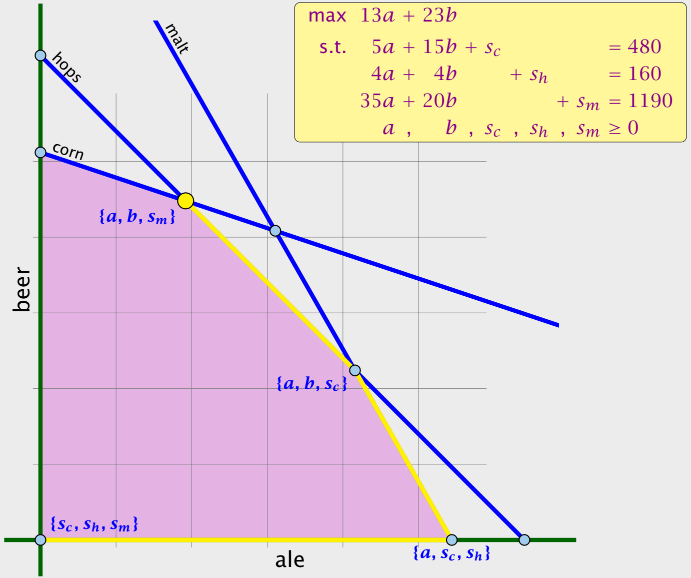

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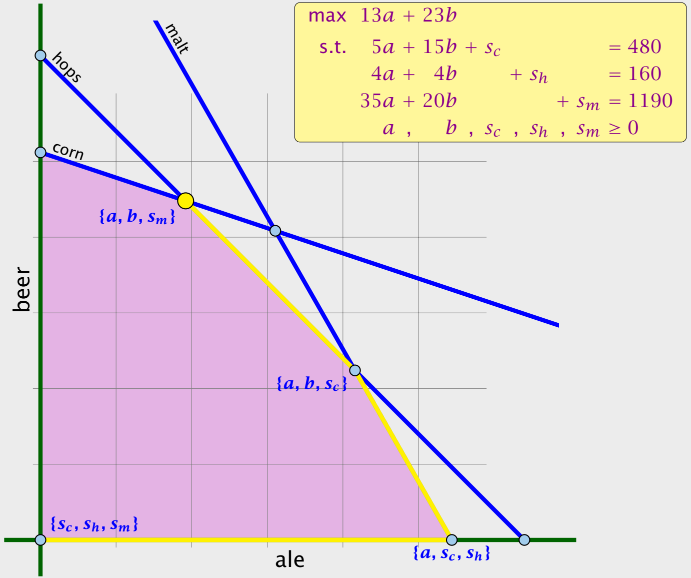

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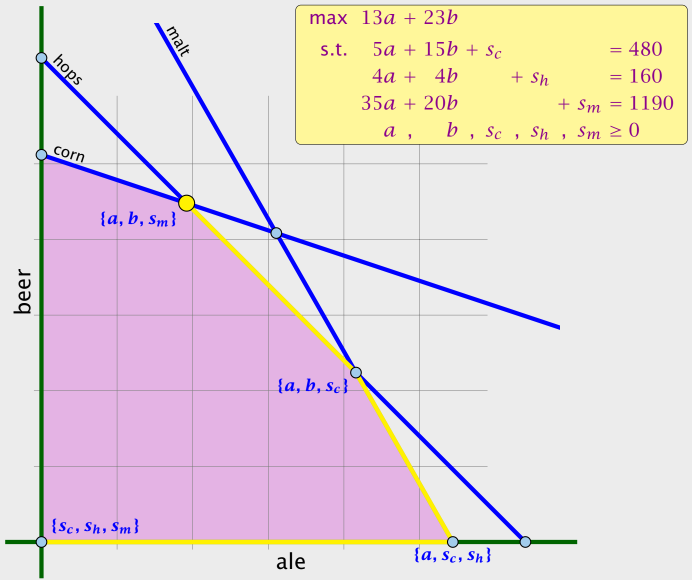

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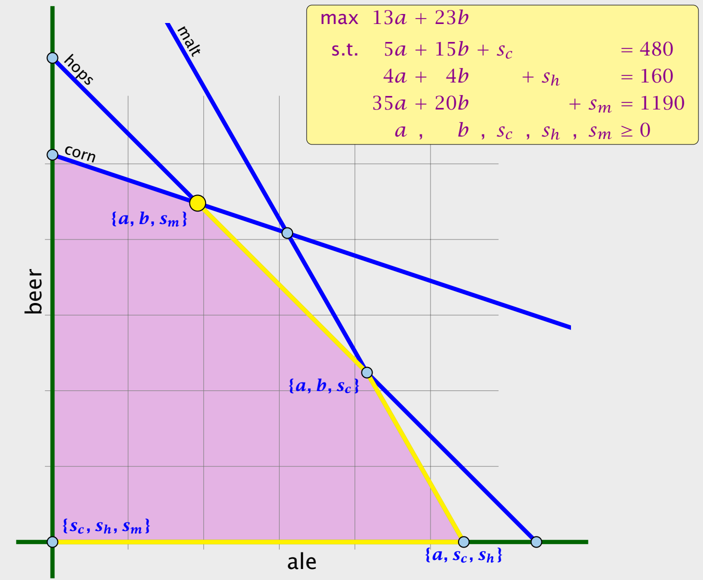

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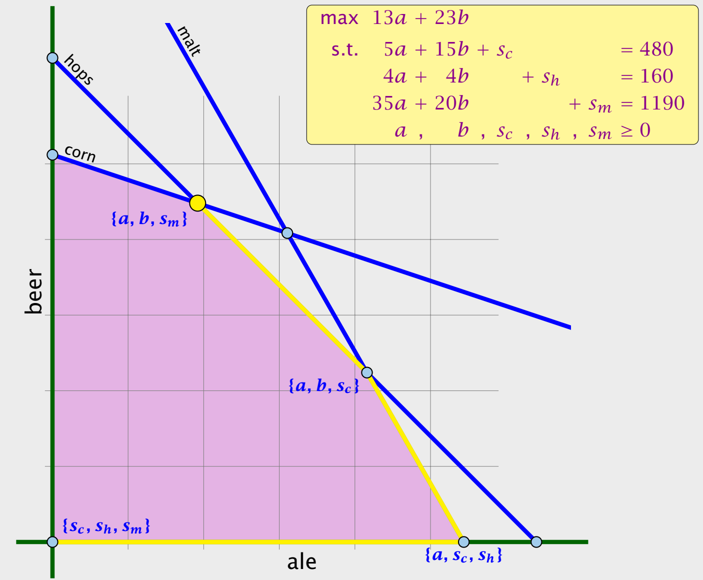

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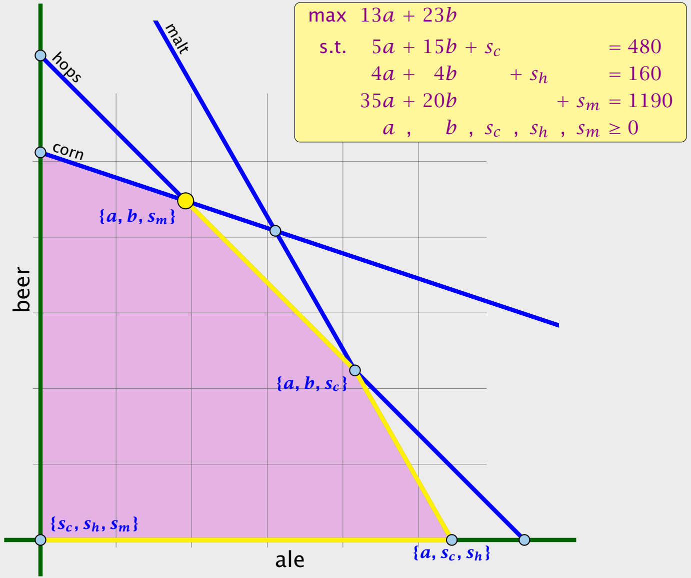

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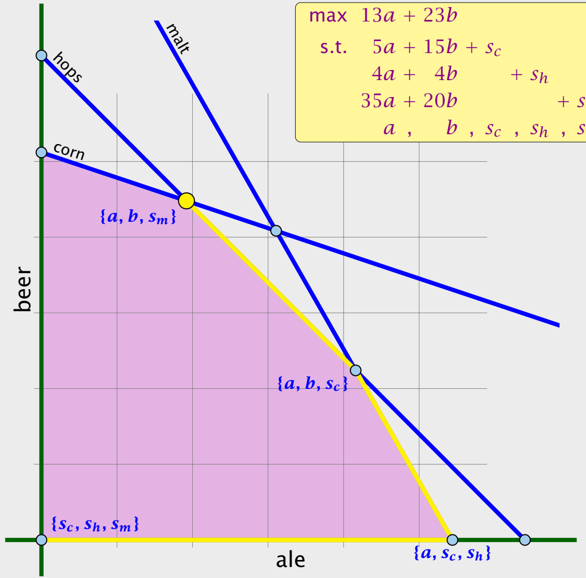

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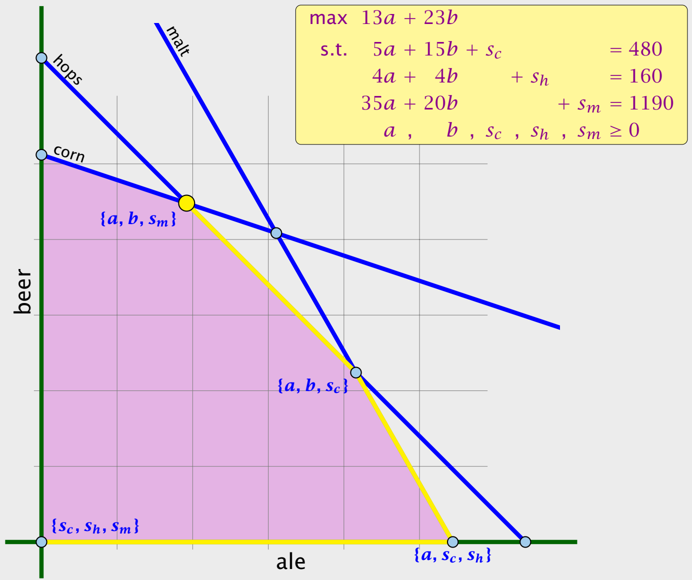

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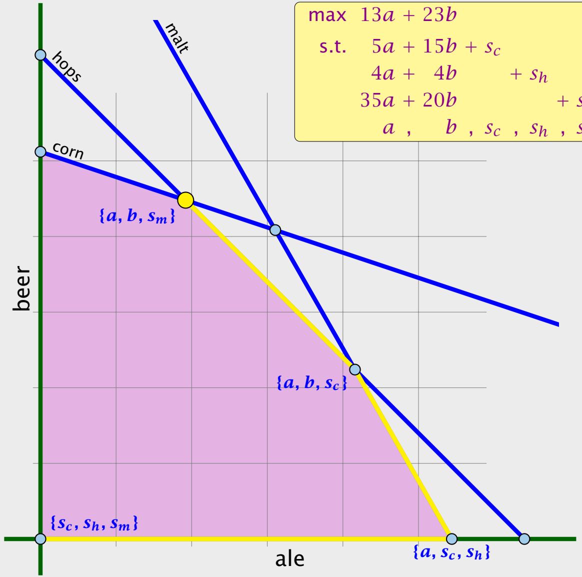

Algebraic Definition of Pivoting

Definition 26 (j-th basis direction)
Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1}A_{*j}$ is called the j-th basis direction for B.

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

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4 Simplex Algorithm

Questions:

Algebraic Definition of Pivoting

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4 Simplex Algorithm

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Algebraic Definition of Pivoting

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \ge & 0 \end{array}$$

The simplex tableaux for basis B is

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The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

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The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

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Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is degenerate (also the basis is degenerate).

Definition 28 (Degeneracy)
A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

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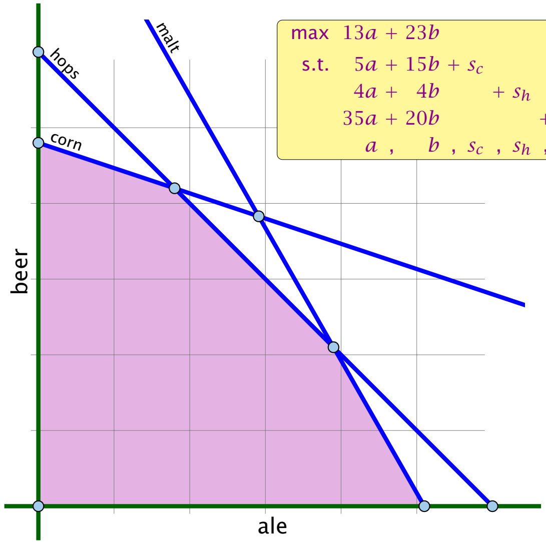

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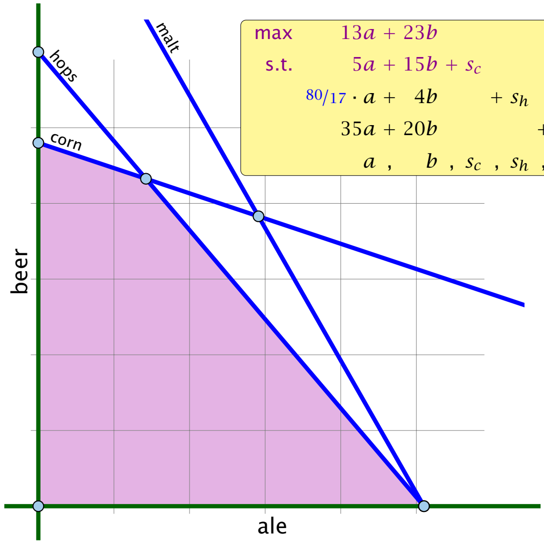

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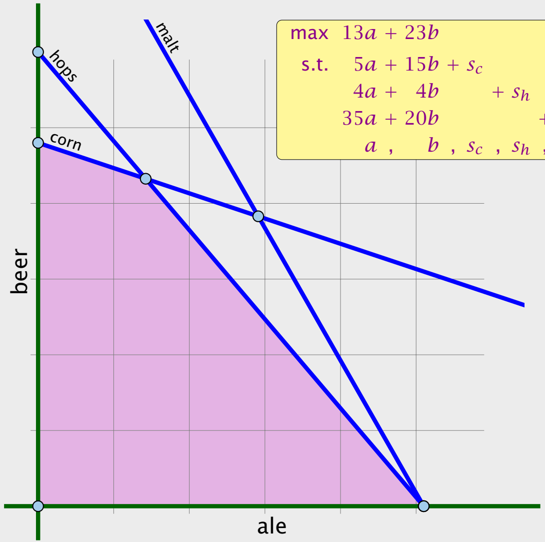

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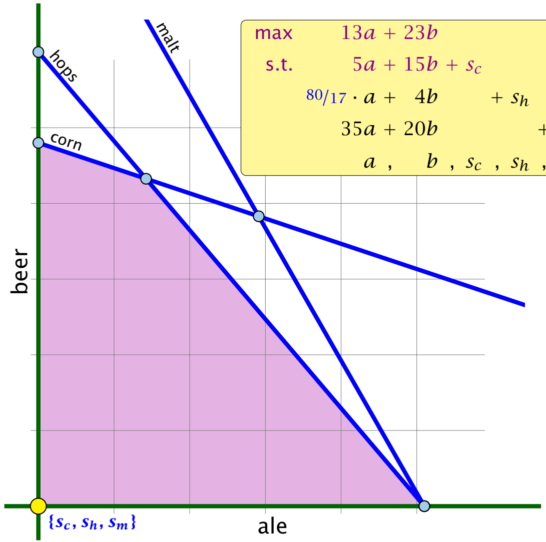

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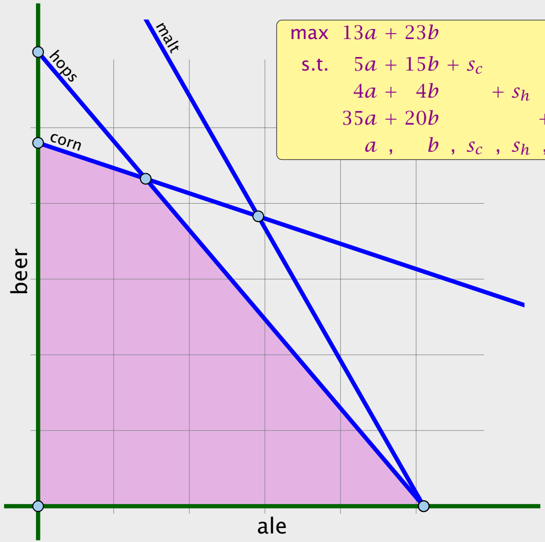

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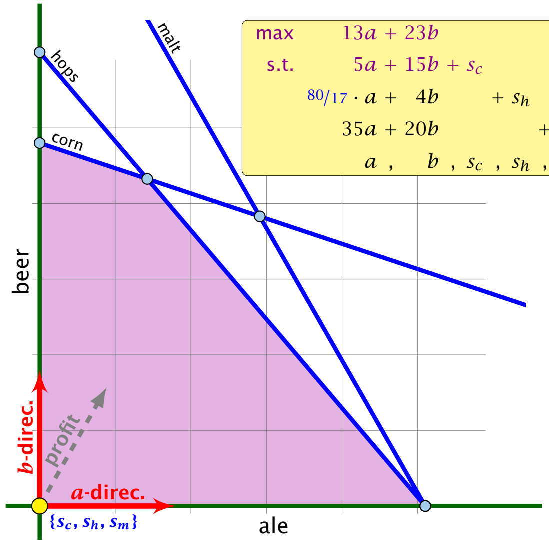

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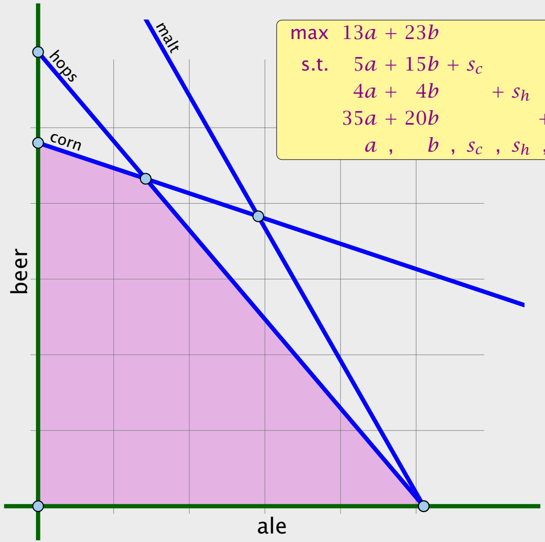

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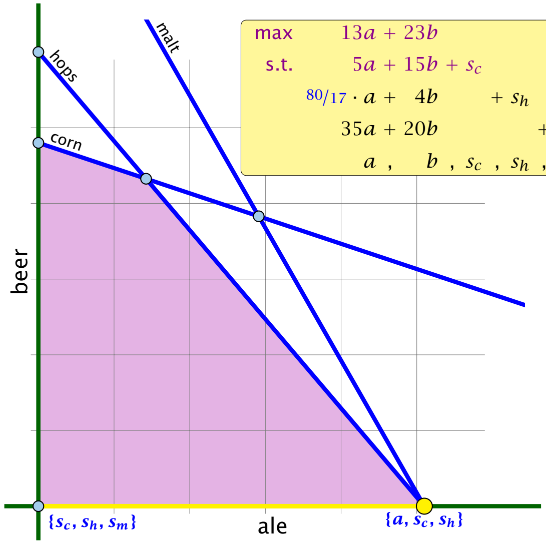

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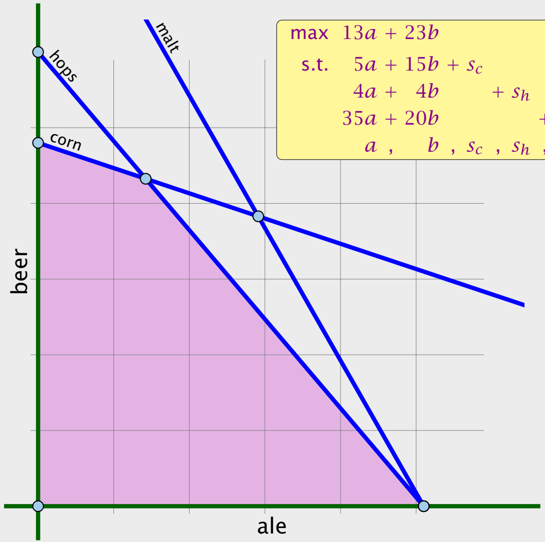

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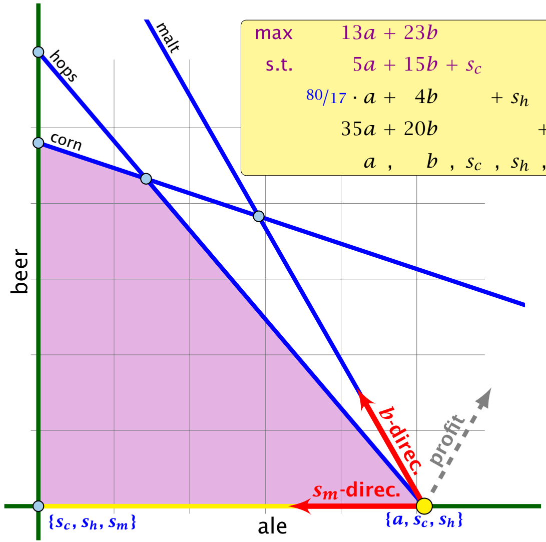

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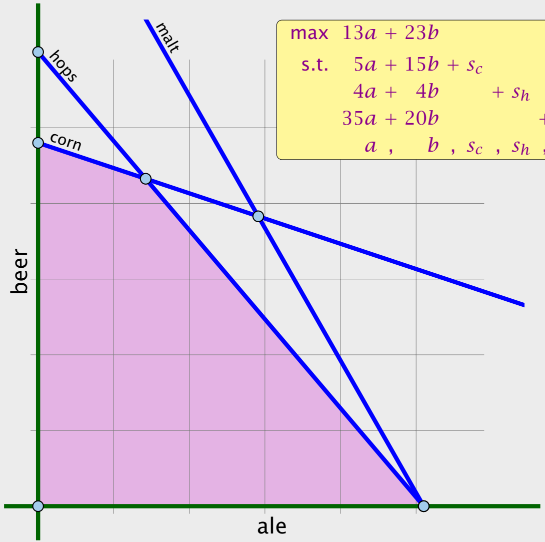

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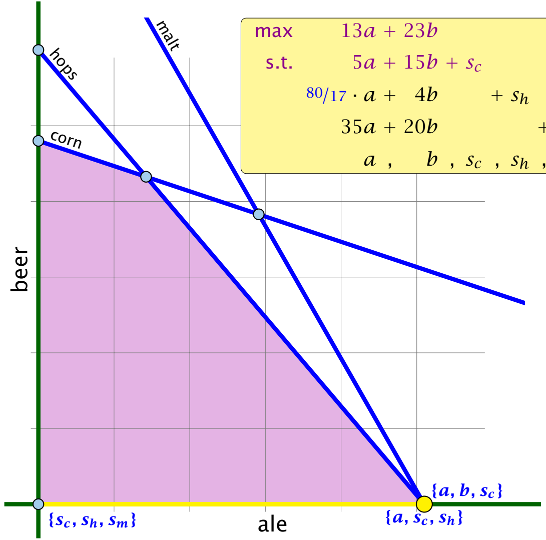

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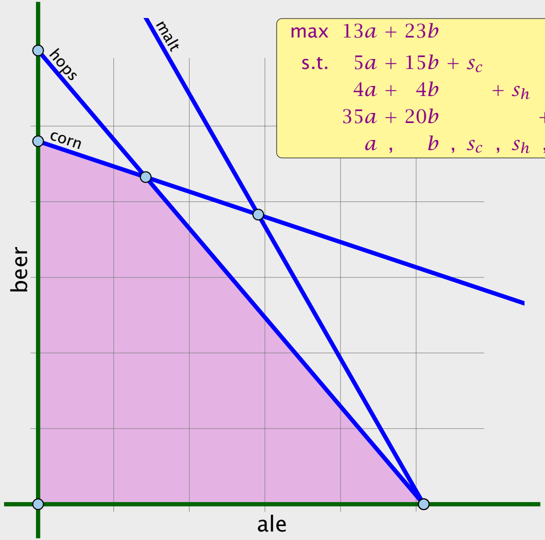

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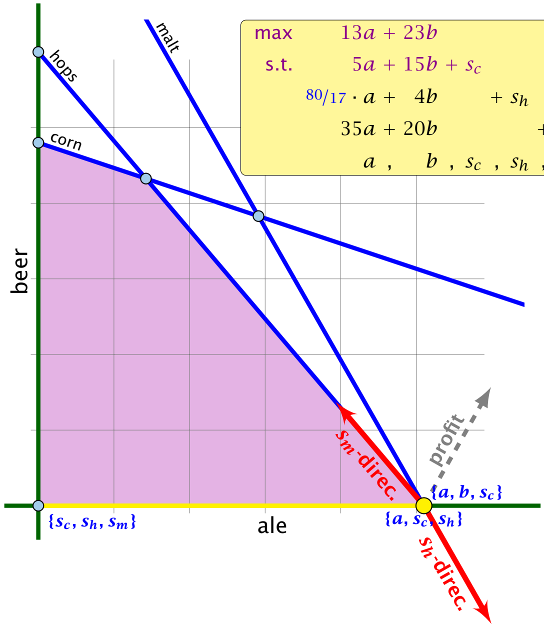

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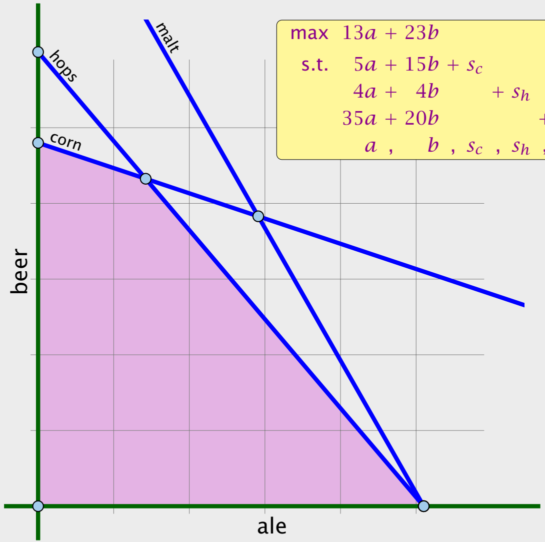

Degenerate Example

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\text{s.t.} & 5a + 15b + s_c & = 480 \\
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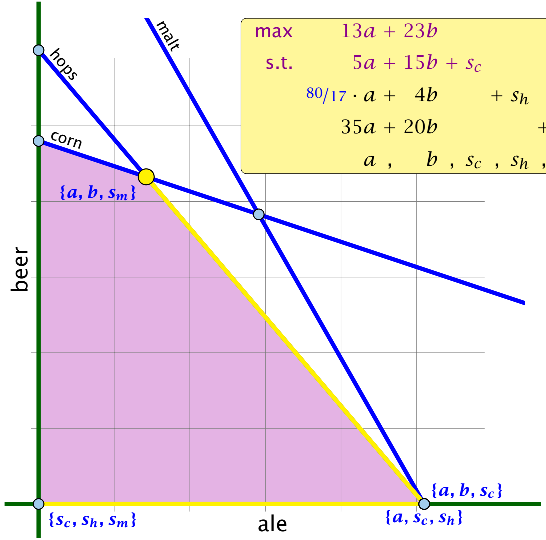

Non Degenerate Example

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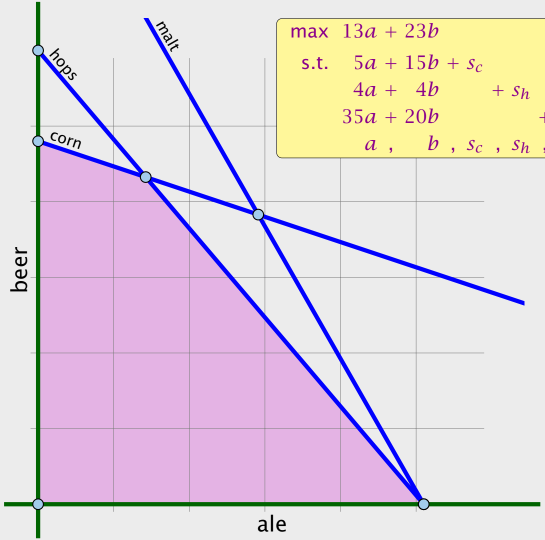

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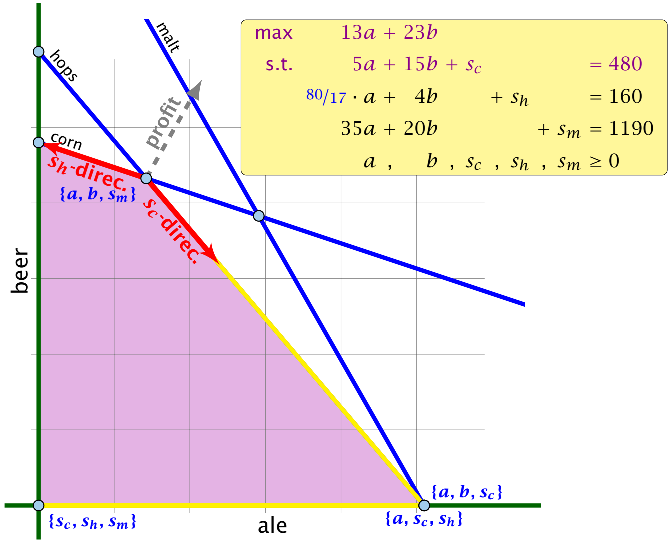

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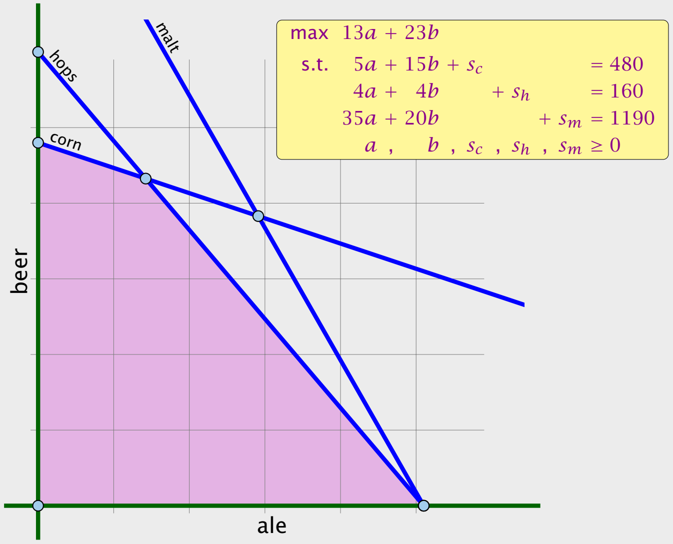

Summary: How to choose pivot-elements

- We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
- The standard choice is the column that maximizes $\tilde{c}_e$.
- If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
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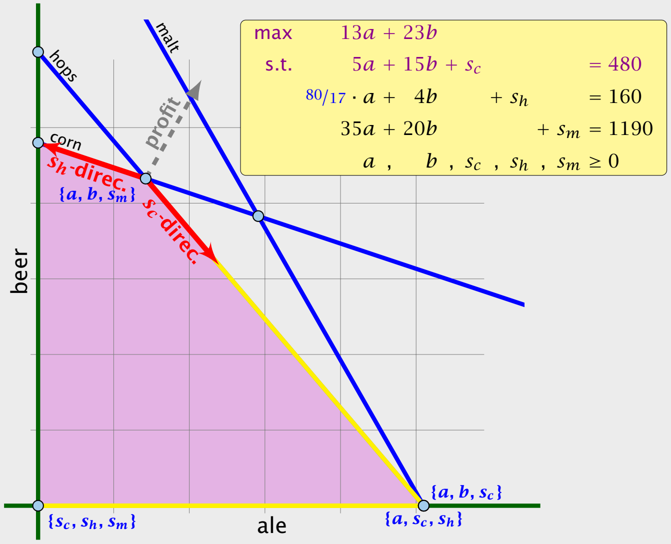

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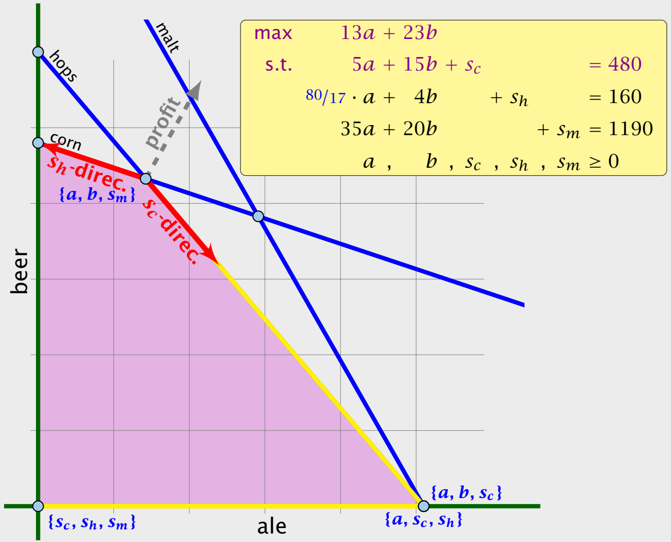

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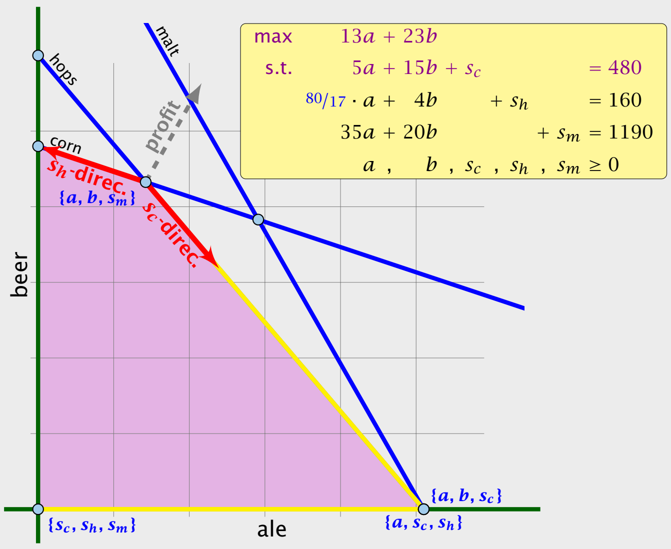

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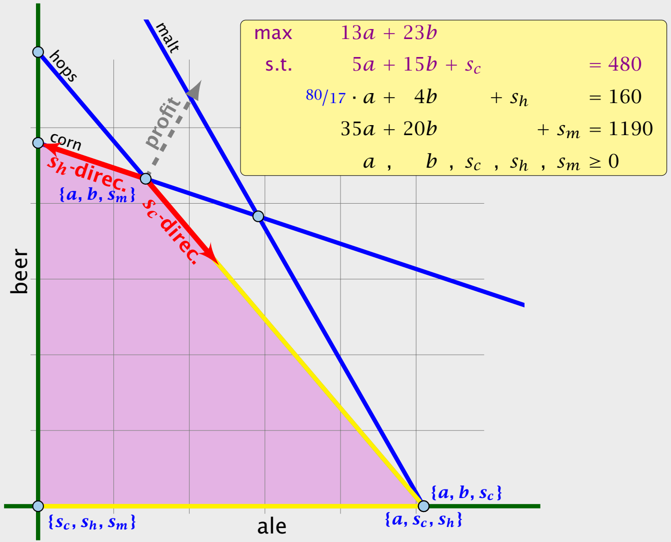

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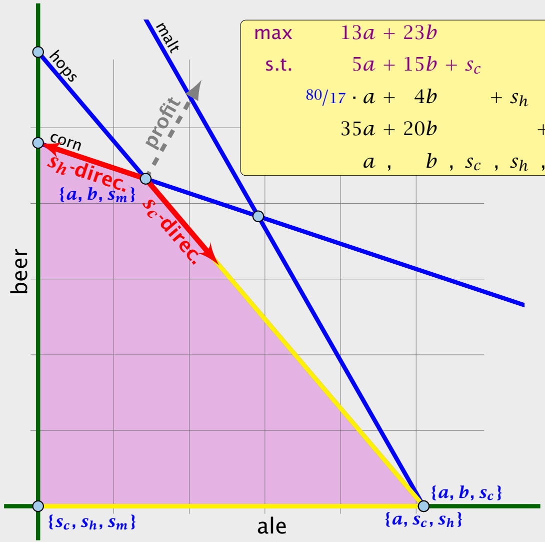

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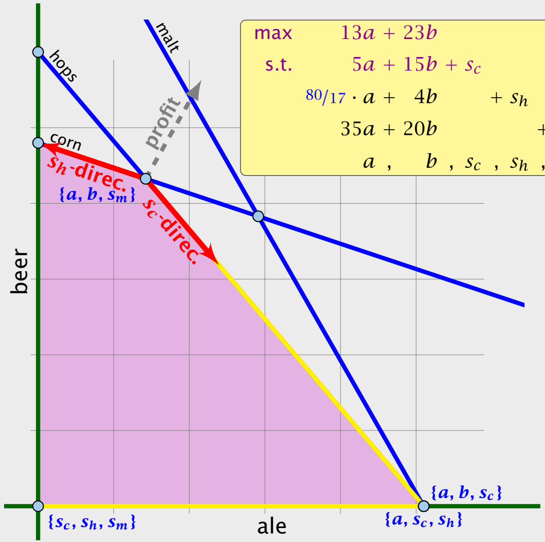

Termination

What do we have so far?
Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

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- $Ax \le b, x \ge 0$, and $\boldsymbol{b} \ge \mathbf{0}$.
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Two phase algorithm

Suppose we want to maximize c^Tx s.t. $Ax = b, x \geq 0$.

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Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \ge 0$.

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Optimality

Lemma 29
Let B be a basis and x^* a BFS corresponding to basis B. $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.

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Duality

How do we get an upper bound to a maximization LP?

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\text{s.t.} & 5a & + & 15b & \leq 480 \\
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\end{array}
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Note that a lower bound is easy to derive. Every choice of $a, b \geq 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \geq 0$) such that $\sum_i y_i a_{ij} \geq c_j$ then $\sum_i y_i b_i$ will be an upper bound.

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Definition 30
Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

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The dual of the dual problem is the primal problem.

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The dual problem is

- $z = -\min\{-c^T x \mid -Ax \geq -b, x \geq 0\}$
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- $z = -\min\{-c^T x \mid -Ax \geq -b, x \geq 0\}$
- $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$

Duality

Definition 30
Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

is called the dual problem.

Duality

Lemma 31
The dual of the dual problem is the primal problem.

Proof:

- $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$
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Weak Duality

Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ and
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x is primal feasible iff $x \in \{x \mid Ax \le b, x \ge 0\}$

y is dual feasible, iff $y \in \{y \mid A^T y \ge c, y \ge 0\}$.

Theorem 32 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^T \hat{x} \le z \le w \le b^T \hat{y} \ .$$

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$A^T\hat{y} \ge c \Rightarrow \hat{x}^T A^T \hat{y} \ge \hat{x}^T c \;(\hat{x} \ge 0)$

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This gives

$$c^T\hat{x} \le \hat{y}^T A\hat{x} \le b^T\hat{y} \;.$$

Since, there exists primal feasible $\hat{x}$ with $c^T\hat{x} = z$, and dual feasible $\hat{y}$ with $b^T y = w$ we get $z \le w$.

If P is unbounded then D is infeasible.

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Weak Duality

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5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

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Proof

Primal:

$$\max\{c^T x \mid Ax = b, x \geq 0\}$$

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Proof

Primal:

$$\max\{c^Tx \mid Ax = b, x \geq 0\}$$
$$= \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\}$$

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\end{aligned}
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Dual:

$$
\min\{[b^T \; -b^T]y \mid [A^T \; -A^T]y \geq c, y \geq 0\}
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Dual:

$$\begin{aligned}&\min\{[b^T \; -b^T]y \mid [A^T \; -A^T]y \geq c, y \geq 0\}\\ &\quad= \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\}\end{aligned}$$

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&\min\{[b^T\ -b^T]y \mid [A^T\ -A^T]y \geq c, y \geq 0\}\\
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Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \ge 0\} \\ &\quad = \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\} \\ &\quad = \max\left\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\right\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \ -b^T] y \mid [A^T \ -A^T] y \ge c, y \ge 0\} \\ &\quad = \min\left\{[b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0\right\} \\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0\right\} \\ &\quad = \min\left\{b^T y' \,\middle|\, A^T y' \ge c\right\} \end{aligned}$$

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Dual:

$$\begin{aligned} &\min\{[b^T \ -b^T] y \mid [A^T \ -A^T] y \geq c, y \geq 0\} \\ &\quad = \min\left\{ [b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0 \right\} \\ &\quad = \min\left\{ b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0 \right\} \\ &\quad = \min\left\{ b^T y' \,\middle|\, A^T y' \geq c \right\} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \geq c$

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5.3 Strong Duality

$P = \max\{c^T x \mid Ax \le b, x \ge 0\}$
n_A: number of variables, m_A: number of constraints

We can put the non-negativity constraints into A (which gives us unrestricted variables): $\bar{P} = \max\{c^T x \mid \bar{A}x \le \bar{b}\}$
$n_{\bar{A}} = n_A$, $m_{\bar{A}} = m_A + n_A$

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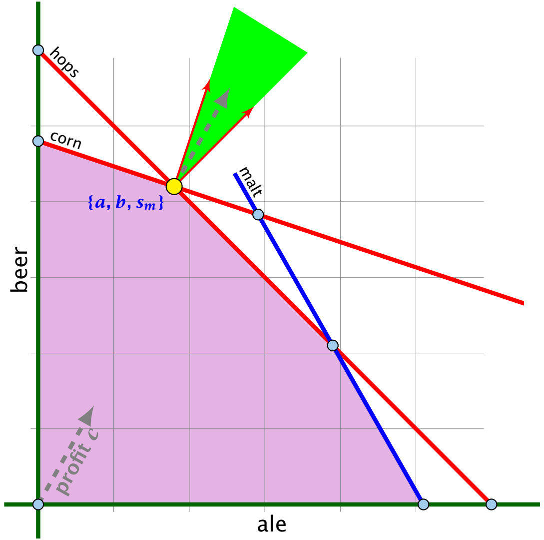

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Strong Duality

Theorem 33 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then

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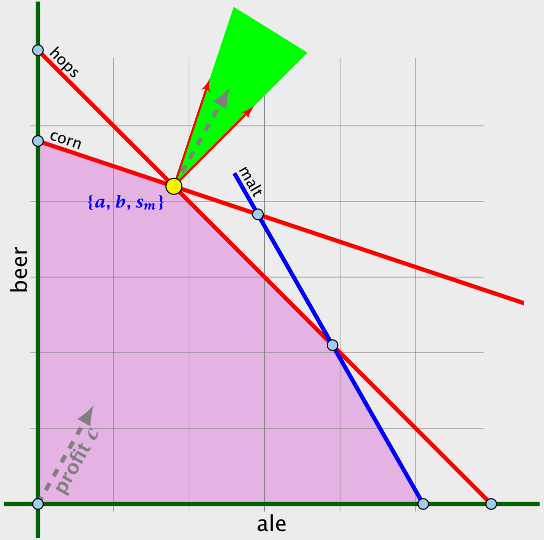

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Lemma 34 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X. Then $\min\{f(x) : x \in X\}$ exists.

(without proof)

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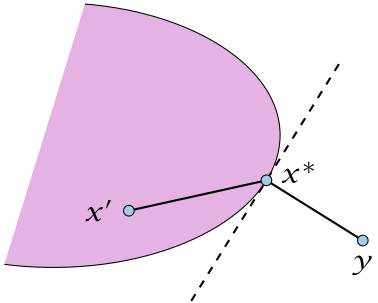
Lemma 35 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y. Moreover for all $x \in X$ we have $(y - x^*)^T(x - x^*) \leq 0$.



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- We want to apply Weierstrass but X may not be bounded.
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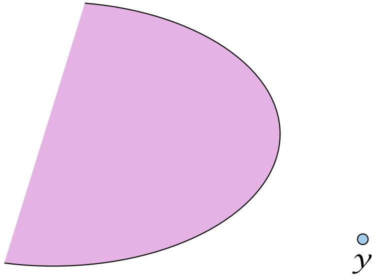

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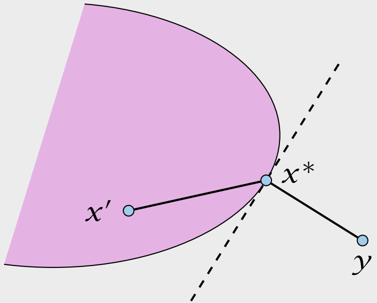

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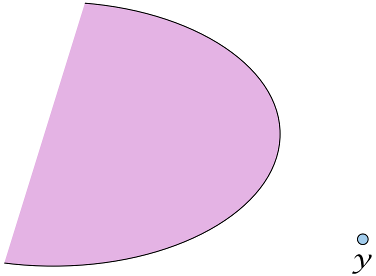

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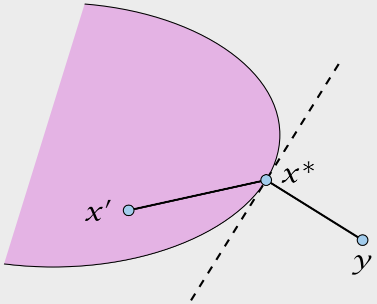

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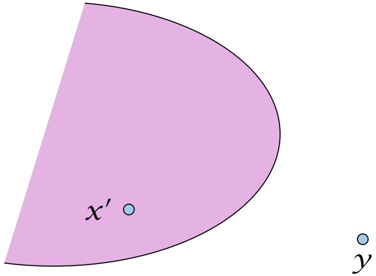

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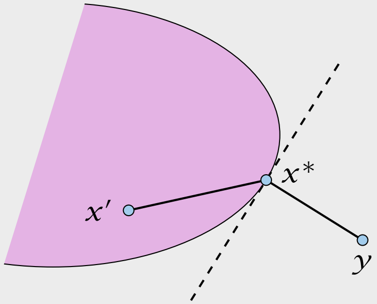

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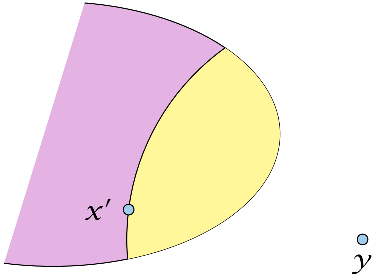

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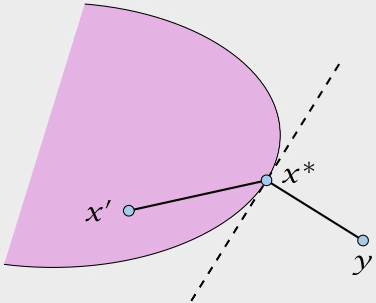

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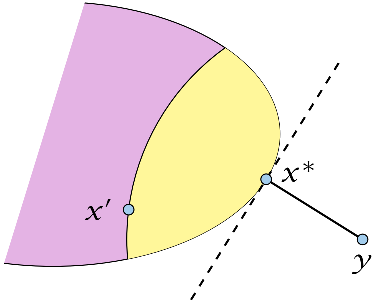

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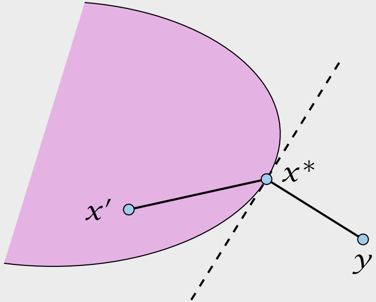

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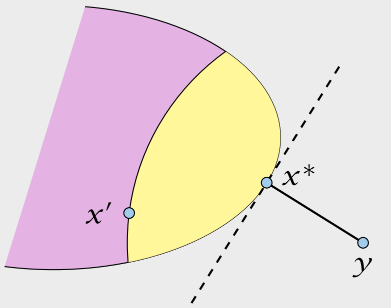

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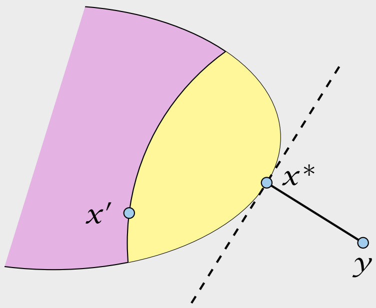

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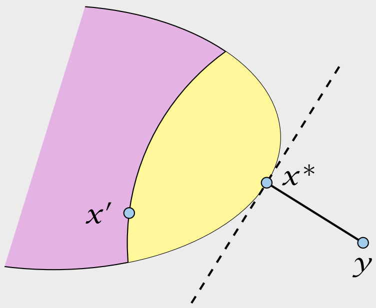

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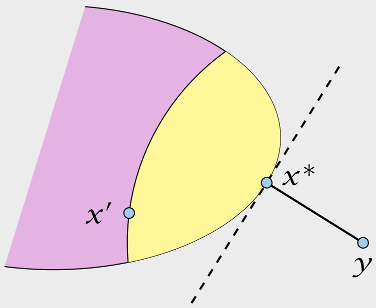

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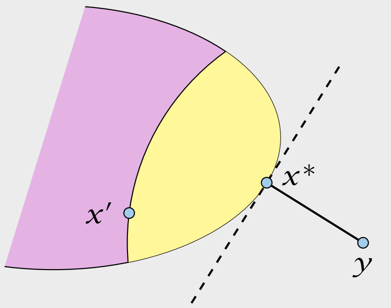

Proof of the Projection Lemma (continued)

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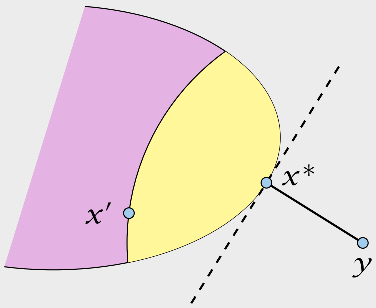

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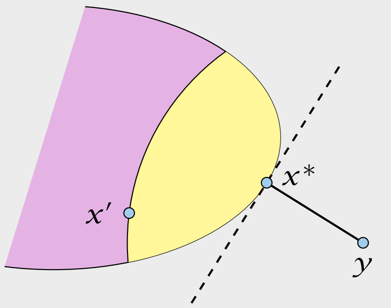

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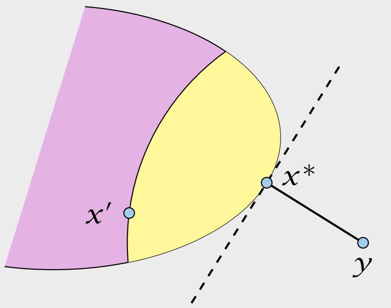

Theorem 36 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^T x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X. ($a^T y < \alpha$; $a^T x \geq \alpha$ for all $x \in X$)

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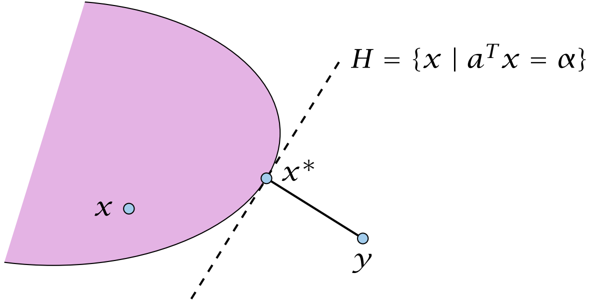

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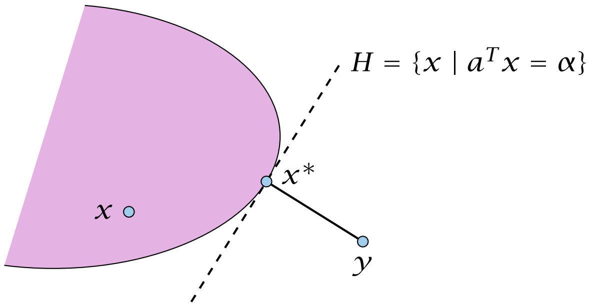

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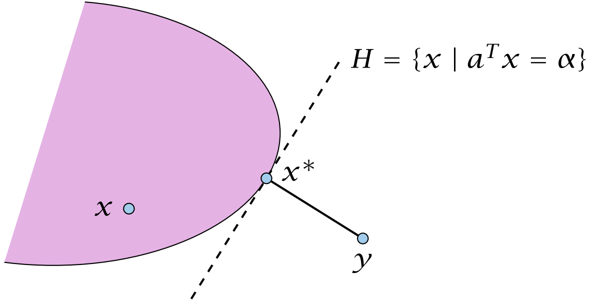

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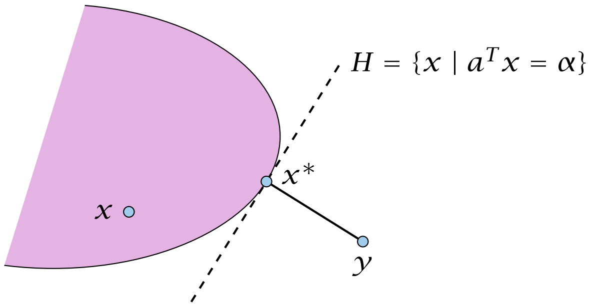

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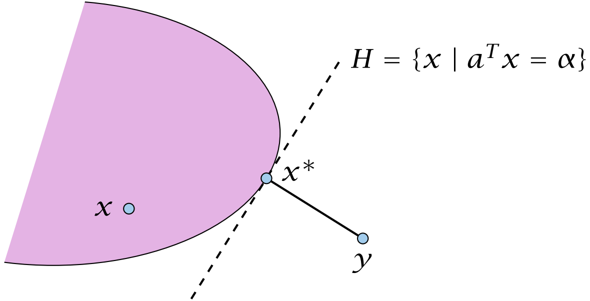

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1. $\exists x \in \mathbb{R}^n$ with $Ax = b$, $x \geq 0$
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Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^T b = y^T A x \geq 0$$

Hence, at most one of the statements can hold.

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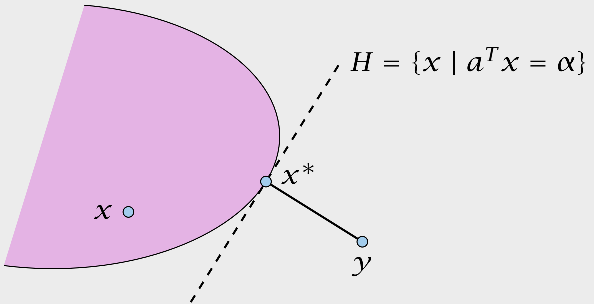

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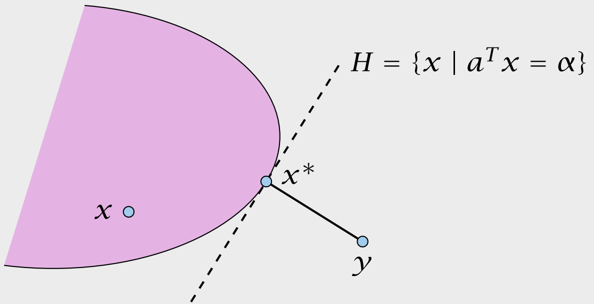

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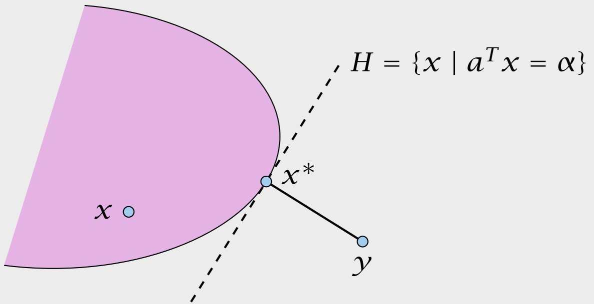

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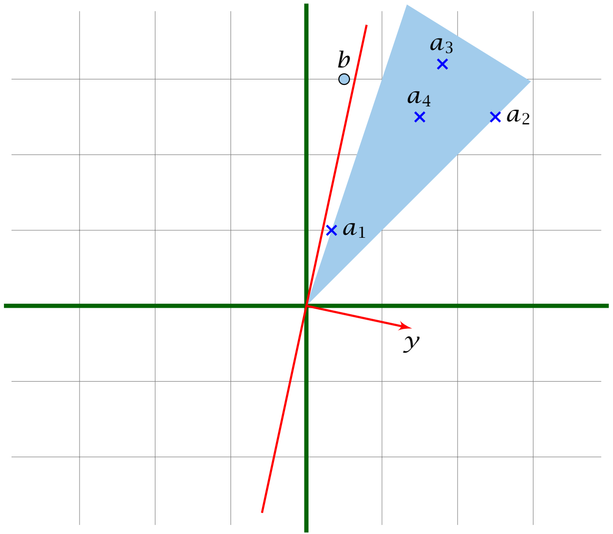

If b is not in the cone generated by the columns of A, there exists a hyperplane y that separates b from the cone.

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Hence, at most one of the statements can hold.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

We want to show that there is y with $A^T y \geq 0$, $b^T y < 0$.

Let y be a hyperplane that separates b from S. Hence, $y^T b < \alpha$ and $y^T s \geq \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$

$y^T Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose x arbitrarily large.

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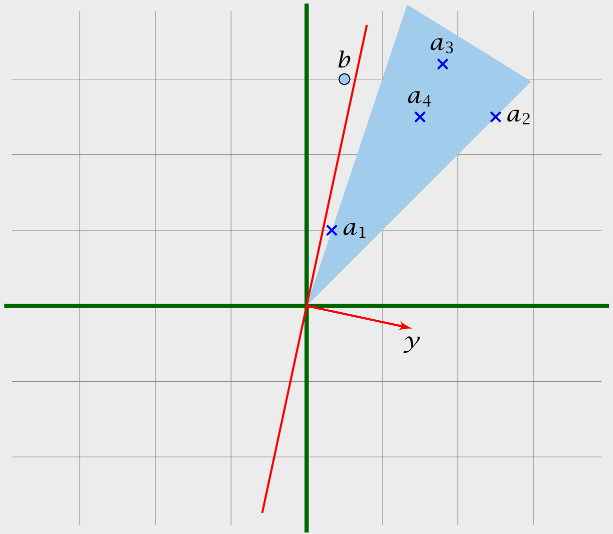

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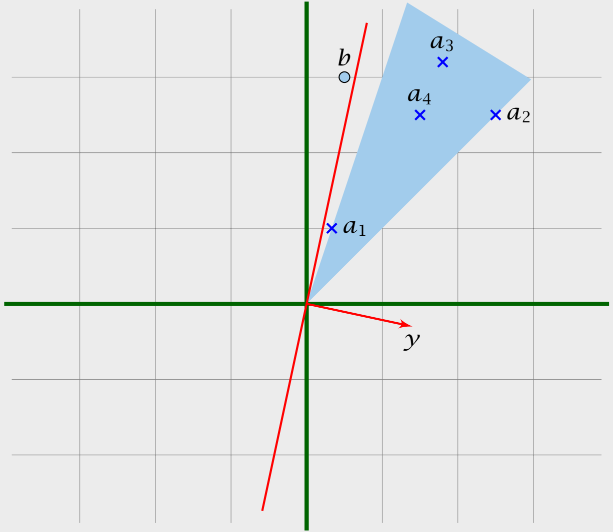

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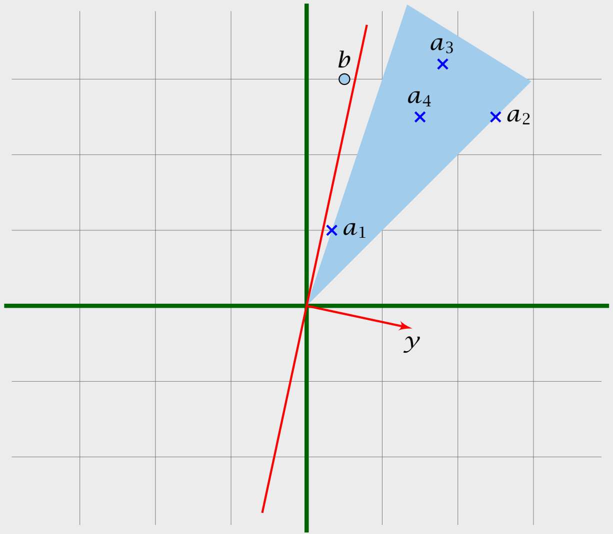

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We want to show that there is y with $A^Ty \geq 0$, $b^Ty < 0$.

Let y be a hyperplane that separates b from S. Hence, $y^Tb < \alpha$ and $y^Ts \geq \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^Tb < 0$

$y^TAx \geq \alpha$ for all $x \geq 0$. Hence, $y^TA \geq 0$ as we can choose x arbitrarily large.

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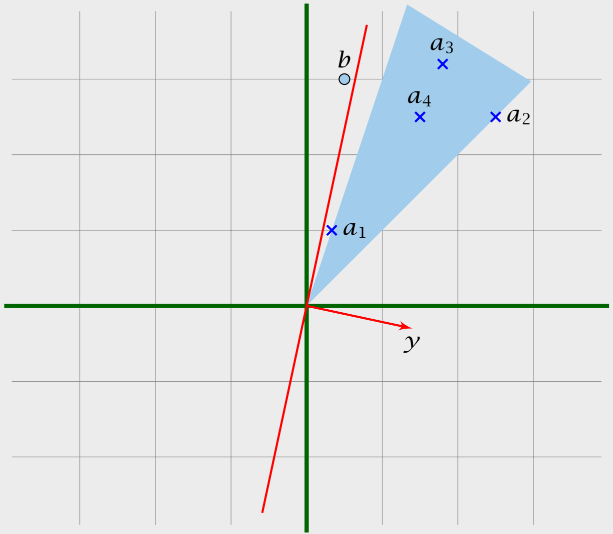

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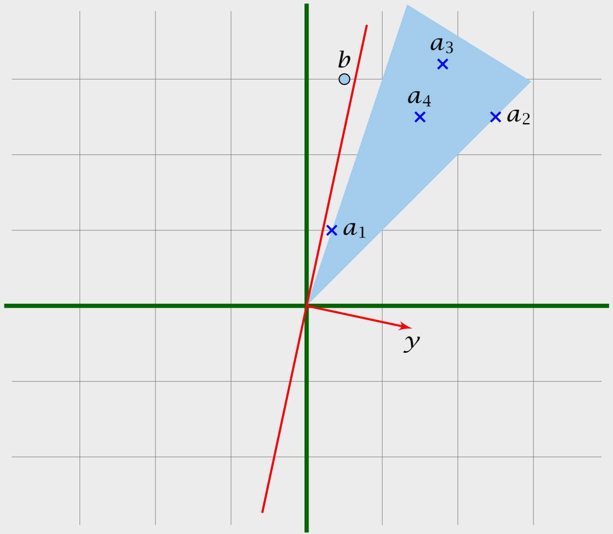

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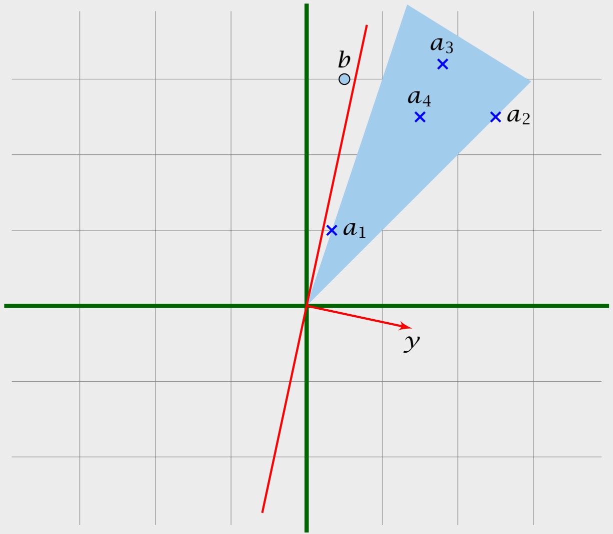

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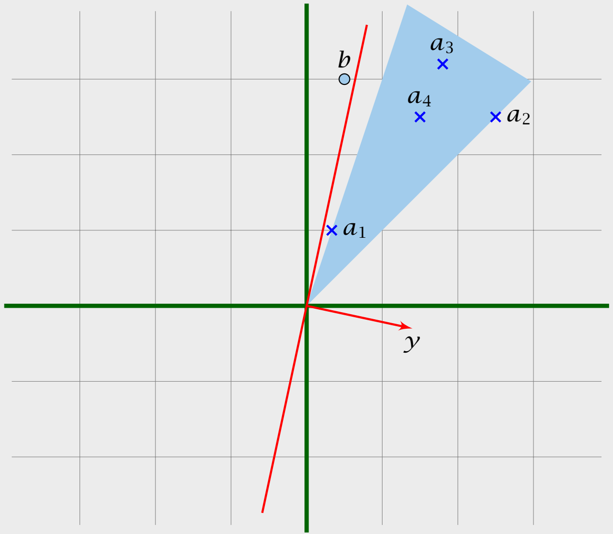

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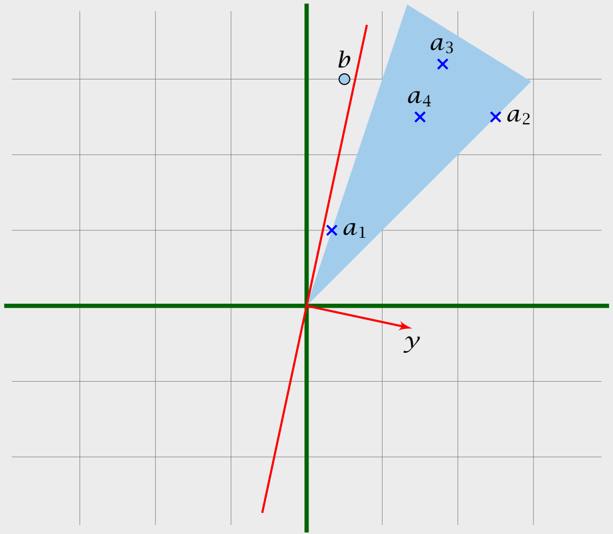

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Lemma 38 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

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Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \geq 0$, $s \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $\begin{bmatrix} A^T \\ I \end{bmatrix} y \geq 0$, $b^T y < 0$

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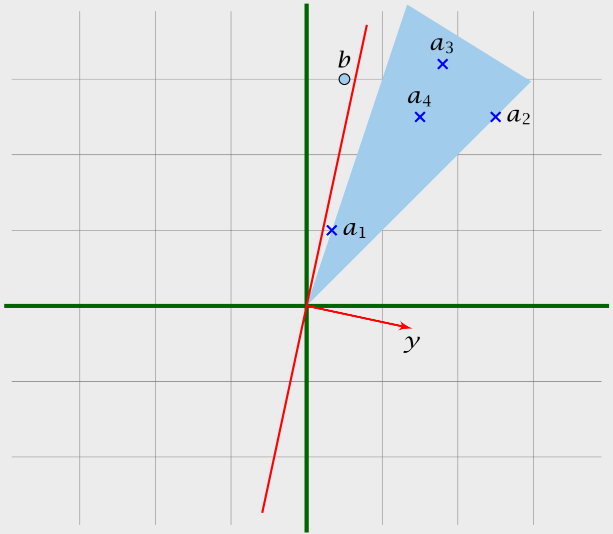

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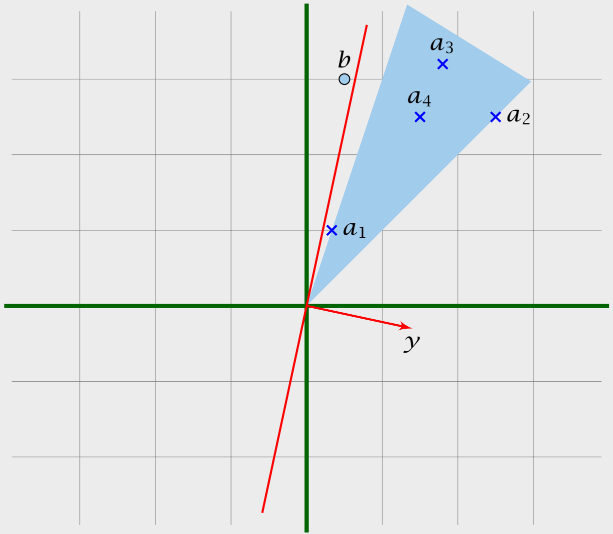
1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A \; I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \geq 0$, $s \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $\begin{bmatrix} A^T \\ I \end{bmatrix} y \geq 0$, $b^T y < 0$

Farkas Lemma



If b is not in the cone generated by the columns of A, there exists a hyperplane y that separates b from the cone.

Proof of Strong Duality

P: $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$

D: $w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$

Theorem 39 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D, respectively (i.e., P and D are non-empty). Then

$$z = w \ .$$

Lemma 38 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ *with* $Ax \le b$, $x \ge 0$
2. $\exists y \in \mathbb{R}^m$ *with* $A^T y \ge 0$, $b^T y < 0$, $y \ge 0$

Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A \ I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \ge 0$, $s \ge 0$
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Proof of Strong Duality

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We show $z < \alpha$ implies $w < \alpha$.

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From the definition of α we know that the first system is infeasible; hence the second must be feasible.

Proof of Strong Duality

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Hence, there exists a solution y, v with $v > 0$.

We can rescale this solution (scaling both y and v) s.t. $v = 1$.

Then y is feasible for the dual but $b^T y < \alpha$. This means that $w < \alpha$.

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Fundamental Questions

Definition 40 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP? yes!
- Is LP in P?

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- Given a primal maximization problem P and a parameter α. Suppose that $\alpha > \mathrm{opt}(P)$.
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- A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost $< \alpha$.

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Complementary Slackness

Lemma 41
Assume a linear program $P = \max\{c^T x \mid Ax \le b; x \ge 0\}$ *has solution* x^* *and its dual* $D = \min\{b^T y \mid A^T y \ge c; y \ge 0\}$ *has solution* y^**.*

1. *If* $x_j^* > 0$ *then the* j*-th constraint in* D *is tight.*
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If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a contraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \le y^{*T} A x^* \le b^T y^*$$

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Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$\begin{array}{llcll} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \leq 480 \\ & 4a & + & 4b & \leq 160 \\ & 35a & + & 20b & \leq 1190 \\ & & & a,b & \geq 0 \end{array}$$

- Entrepeneur: buy resources from brewer at minimum cost
  C, H, M: unit price for corn, hops and malt.

$$\begin{array}{llclcll} \text{min} & 480C & + & 160H & + & 1190M & \\ \text{s.t.} & 5C & + & 4H & + & 35M & \geq 13 \\ & 15C & + & 4H & + & 20M & \geq 23 \\ & & & & & C,H,M & \geq 0 \end{array}$$

  Note that brewer won't sell (at least not all) if e.g. $5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

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Analogous to the proof of weak duality we obtain

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$$c^T x^* \le y^{*T} A x^* \le b^T y^*$$

Because of strong duality we then get

$$c^T x^* = y^{*T} A x^* = b^T y^*$$

This gives e.g.

$$\sum_j (y^T A - c^T)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^T A \ge c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the j-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

Marginal Price:

- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by ε_C, ε_H, and ε_M, respectively.

The profit increases to $\max\{c^Tx \mid Ax \le b+\varepsilon; x \ge 0\}$. Because of strong duality this is equal to

$$\begin{array}{lrcl} \min & (b^T+\epsilon^T)y & & \\ \text{s.t.} & A^Ty & \ge & c \\ & y & \ge & 0 \end{array}$$

Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$\begin{array}{lrcrl} \max & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a,b & \ge 0 \end{array}$$

- Entrepeneur: buy resources from brewer at minimum cost
  C, H, M: unit price for corn, hops and malt.

$$\begin{array}{lrcrcrl} \min & 480C & + & 160H & + & 1190M & \\ \text{s.t.} & 5C & + & 4H & + & 35M & \ge 13 \\ & 15C & + & 4H & + & 20M & \ge 23 \\ & & & & & C,H,M & \ge 0 \end{array}$$

Note that brewer won't sell (at least not all) if e.g. $5C+4H+35M < 13$ as then brewing ale would be advantageous.

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If ϵ is "small" enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

- If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
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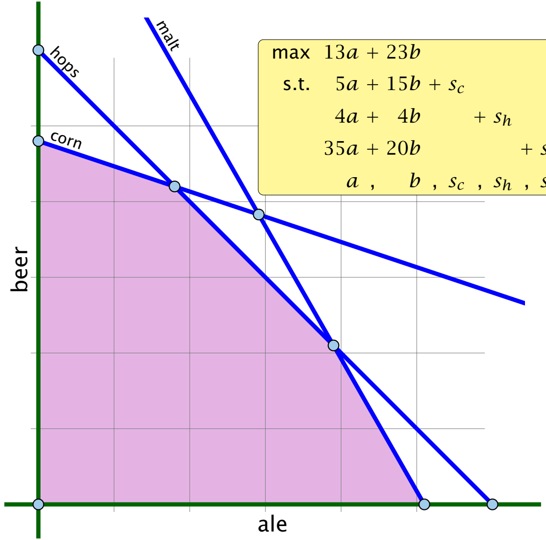

$$\begin{array}{lrcrcrcrcrcl}
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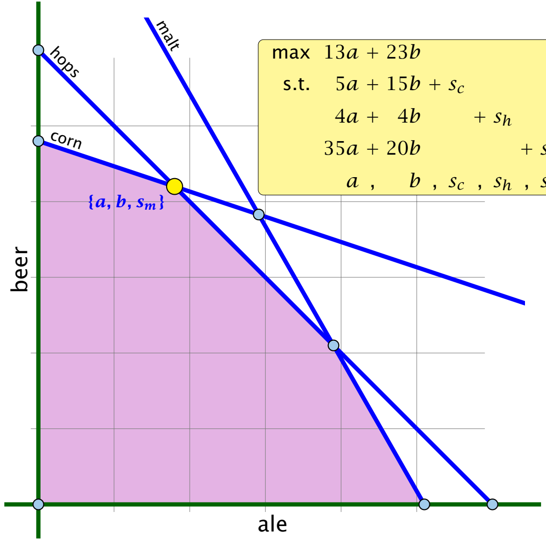

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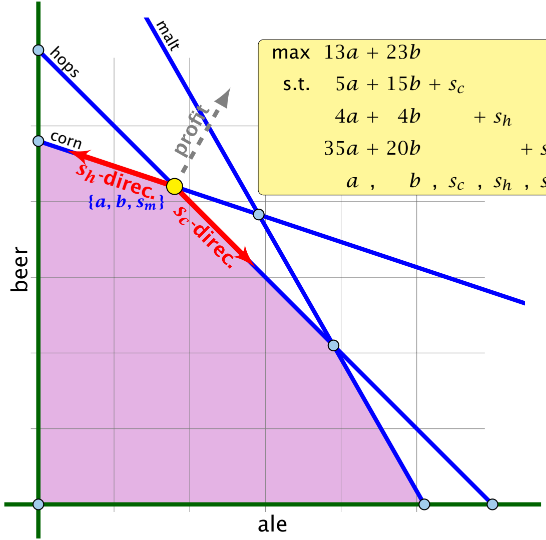

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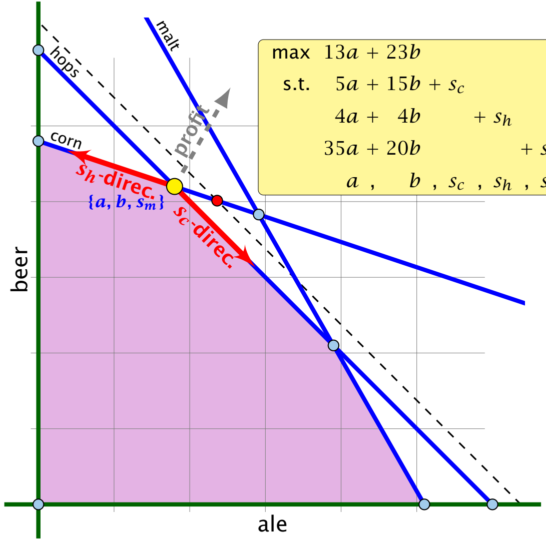

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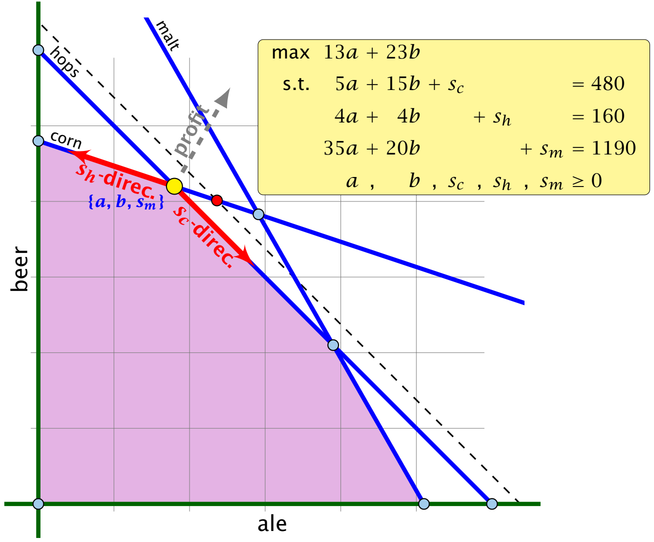

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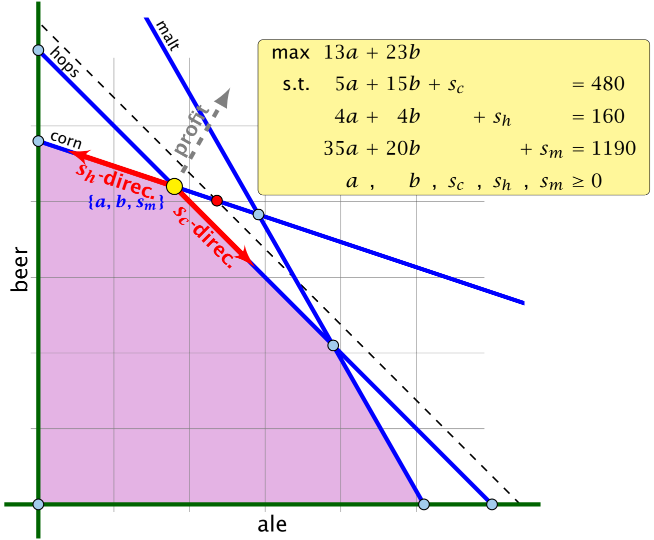

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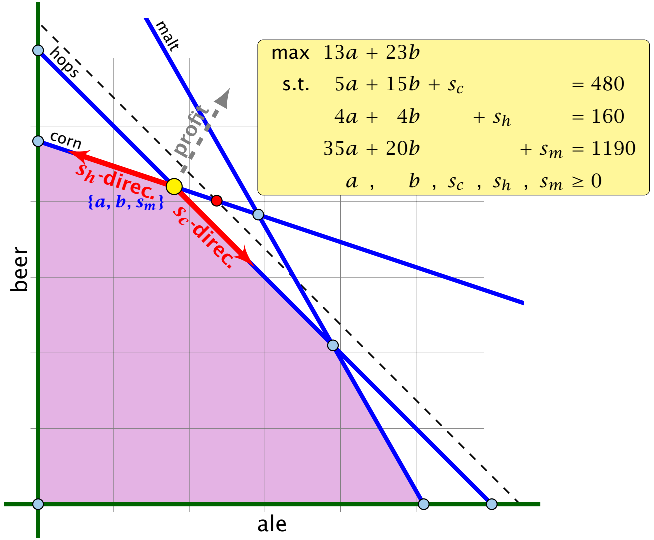

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Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

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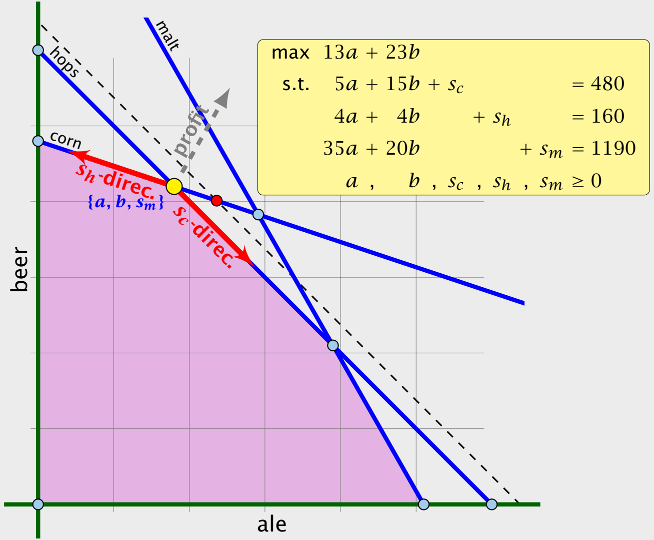

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Flows

Definition 42

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)

$$0 \le f_{xy} \le c_{xy} \ .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$

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Maximum Flow Problem:

Find an (s,t)-flow with maximum value.

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LP-Formulation of Maxflow

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\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
& \forall w \ne s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
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$$

$$
\begin{array}{lllcr}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \ne s,t): & 1\ell_{sy} \qquad + 1p_y & \ge & 1 \\
 & f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
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 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + 0 & \geq & 0 \\
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\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \leq & c_{zw} & \ell_{zw} \\
 & \forall w \neq s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
 & & f_{zw} & \geq & 0 &
\end{array}
$$

$$
\begin{array}{lllcr}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} + 1p_y & \geq & 1 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x & \geq & -1 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x & \geq & 0 \\
 & f_{st}: & 1\ell_{st} & \geq & 1 \\
 & f_{ts}: & 1\ell_{ts} & \geq & -1 \\
 & & \ell_{xy} & \geq & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y\neq s,t): & 1\ell_{xy}-1p_x+1p_y & \geq & 0\\
& f_{sy}\ (y\neq s,t): & 1\ell_{sy}-\ p_s+1p_y & \geq & 0\\
& f_{xs}\ (x\neq s,t): & 1\ell_{xs}-1p_x+\ p_s & \geq & 0\\
& f_{ty}\ (y\neq s,t): & 1\ell_{ty}-\ p_t+1p_y & \geq & 0\\
& f_{xt}\ (x\neq s,t): & 1\ell_{xt}-1p_x+\ p_t & \geq & 0\\
& f_{st}: & 1\ell_{st}-\ p_s+\ p_t & \geq & 0\\
& f_{ts}: & 1\ell_{ts}-\ p_t+\ p_s & \geq & 0\\
& & \ell_{xy} & \geq & 0
\end{array}
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with $p_t = 0$ and $p_s = 1$.

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& f_{sy}\ (y\neq s,t): & 1\ell_{sy}-\ 1+1p_y & \geq & 0\\
& f_{xs}\ (x\neq s,t): & 1\ell_{xs}-1p_x+\ 1 & \geq & 0\\
& f_{ty}\ (y\neq s,t): & 1\ell_{ty}-\ 0+1p_y & \geq & 0\\
& f_{xt}\ (x\neq s,t): & 1\ell_{xt}-1p_x+\ 0 & \geq & 0\\
& f_{st}: & 1\ell_{st}-\ 1+\ 0 & \geq & 0\\
& f_{ts}: & 1\ell_{ts}-\ 0+\ 1 & \geq & 0\\
& & \ell_{xy} & \geq & 0
\end{array}
$$

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\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & & \ell_{xy} & \geq & 0 \\
 & & p_s & = & 1 \\
 & & p_t & = & 0
\end{array}
$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

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 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - p_t + 1p_y & \geq & 0 \\
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$$\begin{array}{llrcl} \min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\ \text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\ & & \ell_{xy} & \geq & 0 \\ & & p_s & = & 1 \\ & & p_t & = & 0 \end{array}$$

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with $p_t = 0$ and $p_s = 1$.

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.

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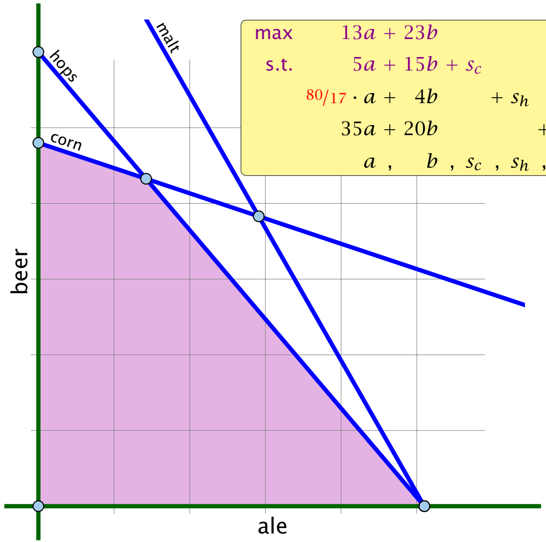
Degeneracy Revisited

Degeneracy Revisited

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

Degenerate Example

$$
\begin{array}{llll}
\text{max} & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & & = 480 \\
 & \tfrac{80}{17} \cdot a + 4b & + s_h & = 160 \\
 & 35a + 20b & + s_m & = 1190 \\
 & a\ ,\quad b\ ,\ s_c\ ,\ s_h\ ,\ s_m \geq 0 & &
\end{array}
$$



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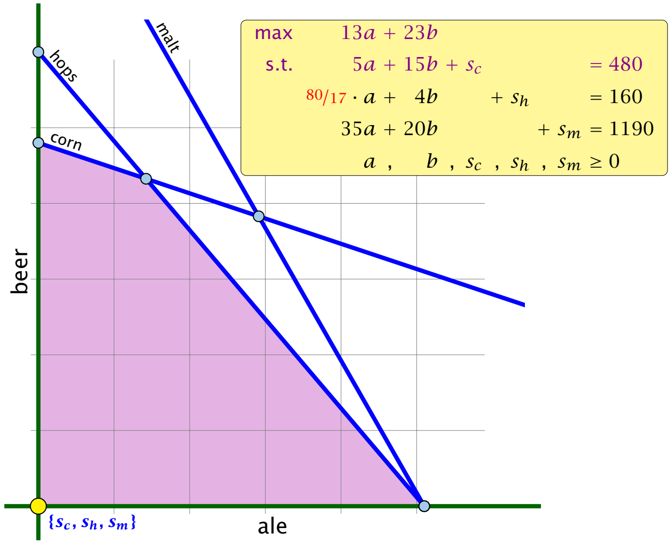

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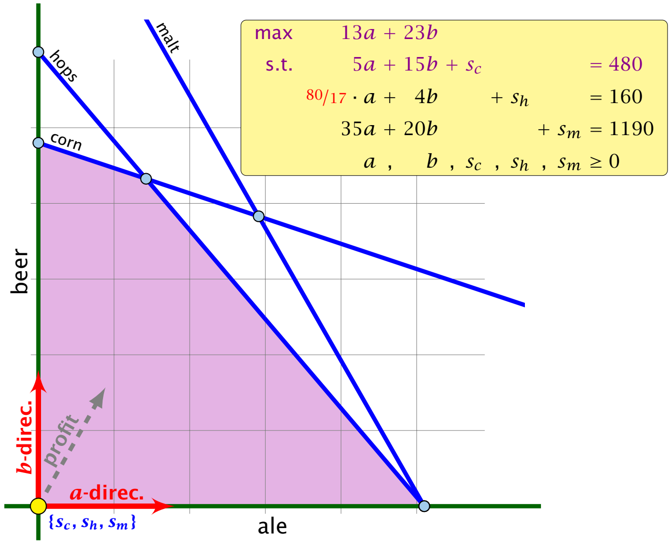

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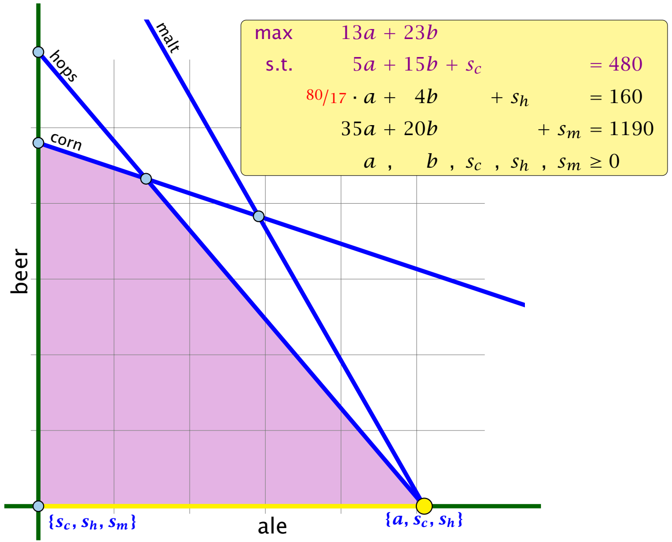

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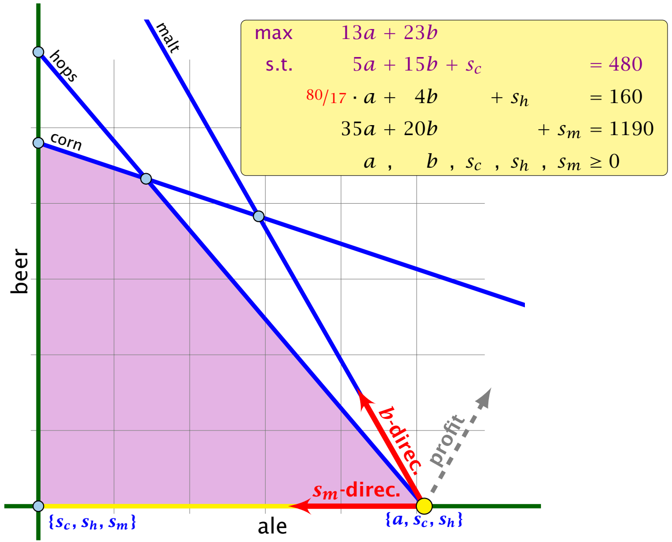

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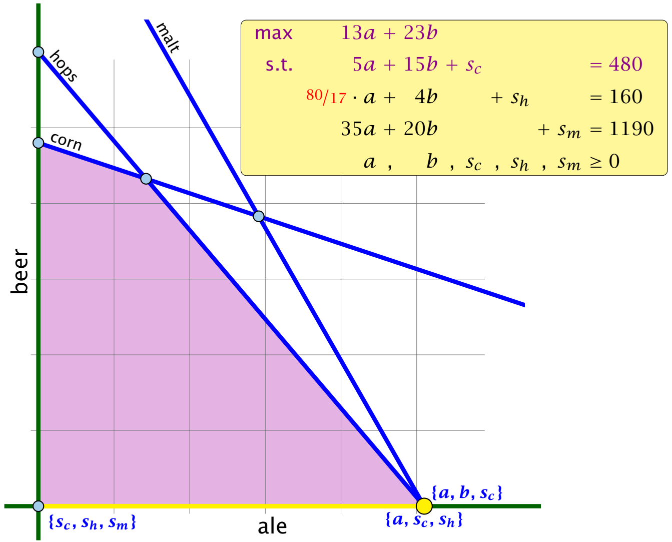

$$
\begin{array}{llrcrcrcrcrcl}
\text{max} & & 13a & + & 23b & & & & & & & & \\
\text{s.t.} & & 5a & + & 15b & + & s_c & & & & & = & 480 \\
& {}^{80}\!/_{17} \cdot & a & + & 4b & & & + & s_h & & & = & 160 \\
& & 35a & + & 20b & & & & & + & s_m & = & 1190 \\
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\end{array}
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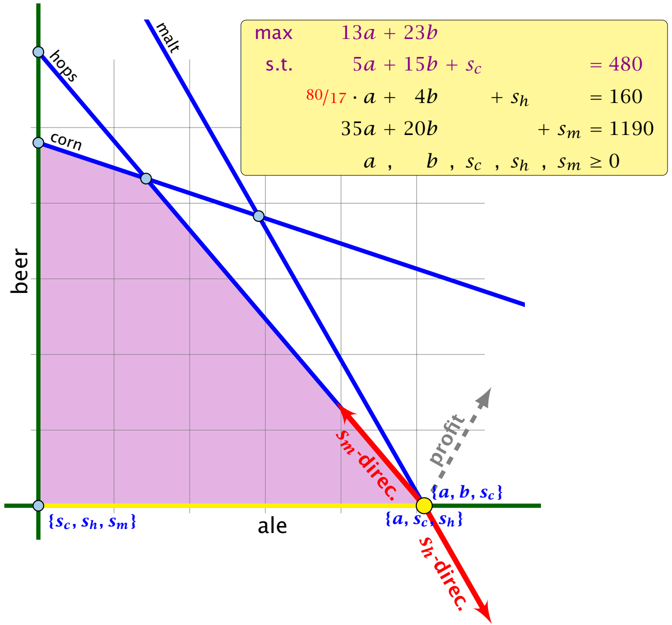

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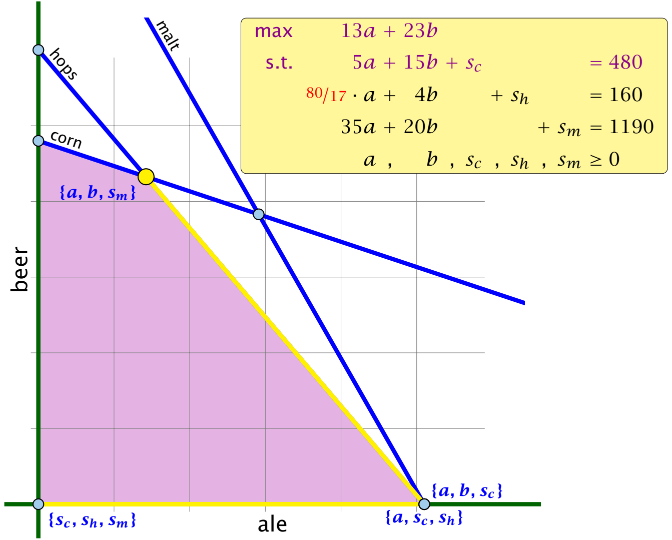

$$
\begin{array}{lrcrcrcrcrcl}
\text{max} & 13a & + & 23b & & & & & & & & \\
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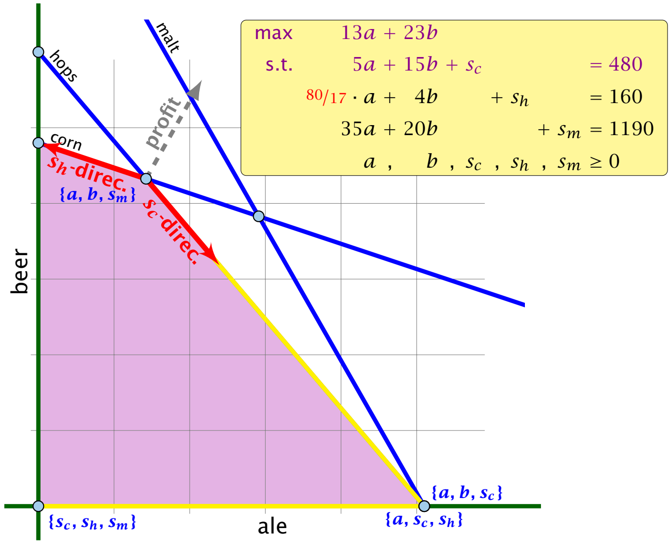

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If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

Idea:

Given feasible $\mathrm{LP} := \max\{c^Tx, Ax = b; x \geq 0\}$. Change it into $\mathrm{LP}' := \max\{c^Tx, Ax = b', x \geq 0\}$ such that

1. LP' is feasible
2. If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1}b \not\geq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).
3. LP' has no degenerate basic solutions

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\begin{array}{lrcrcrcrcrcl}
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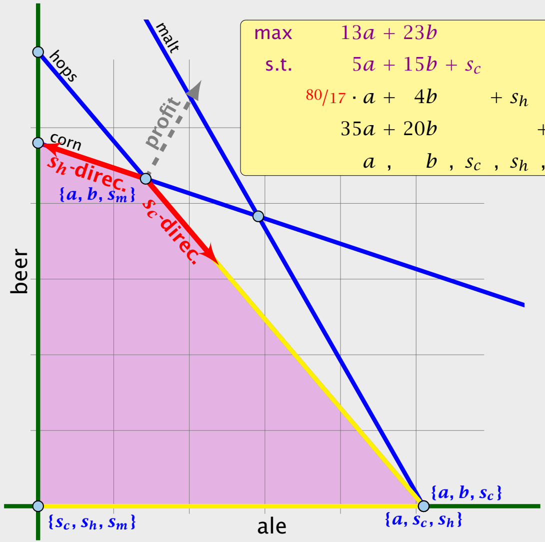

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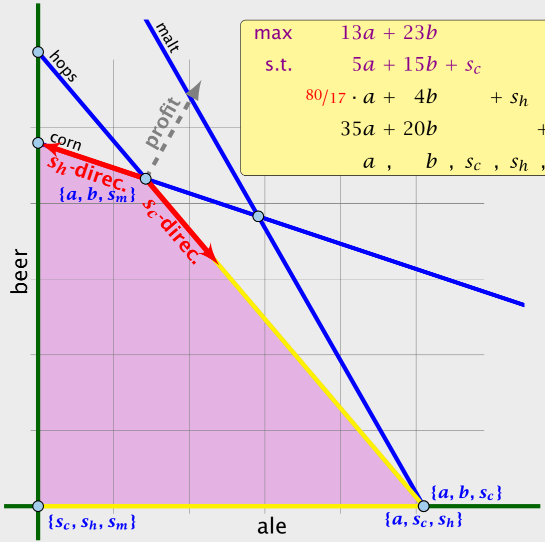

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III. LP' has no degenerate basic solutions

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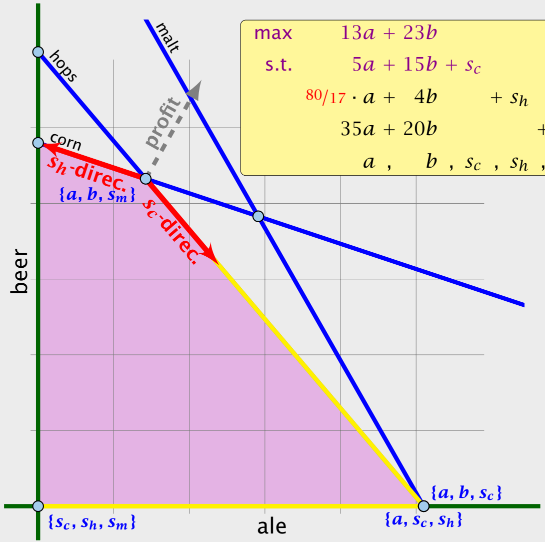

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If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

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Given feasible $\mathrm{LP} := \max\{c^T x, Ax = b; x \geq 0\}$. Change it into $\mathrm{LP}' := \max\{c^T x, Ax = b', x \geq 0\}$ such that

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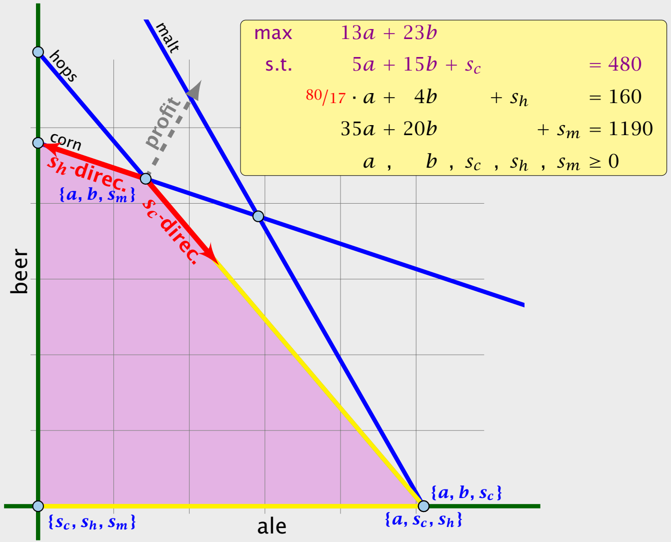

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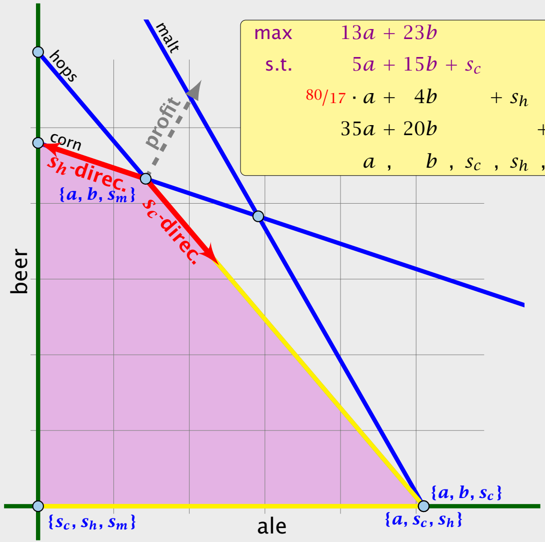
Degenerate Example



$$\begin{array}{lrcl} \max & 13a + 23b & & \\ \text{s.t.} & 5a + 15b + s_c & = & 480 \\ & \tfrac{80}{17} \cdot a + 4b + s_h & = & 160 \\ & 35a + 20b + s_m & = & 1190 \\ & a, b, s_c, s_h, s_m & \geq & 0 \end{array}$$

Perturbation

Let B be index set of some basis with basic solution

$$x_B^* = A_B^{-1}b \geq 0, x_N^* = 0 \quad \text{(i.e. } B \text{ is feasible)}$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \quad \text{for } \varepsilon > 0 \ .$$

This is the perturbation that we are using.

Degeneracy Revisited

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

Idea:
Given feasible $\text{LP} := \max\{c^T x, Ax = b; x \geq 0\}$. Change it into $\text{LP}' := \max\{c^T x, Ax = b', x \geq 0\}$ such that

- **I.** LP' is feasible
- **II.** If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1}b \ngeq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).
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III. LP' has no degenerate basic solutions

Property I

The new LP is feasible because the set B of basis variables provides a feasible basis:

$$A_B^{-1}\left(b + A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right) = x_B^* + \begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix} \geq 0 \ .$$

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Property II

Let $\tilde{B}$ be a non-feasible basis. This means $(A_{\tilde{B}}^{-1}b)_i < 0$ for some row i.

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Let $\tilde{B}$ be a non-feasible basis. This means $(A_{\tilde{B}}^{-1}b)_i < 0$ for some row i.

Then for small enough $\epsilon > 0$

$$\left(A_{\tilde{B}}^{-1}\left(b + A_B\begin{pmatrix}\varepsilon\\ \vdots \\ \varepsilon^m\end{pmatrix}\right)\right)_i$$

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Hence, $\tilde{B}$ is not feasible.

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Property III

Let $\tilde{B}$ be a basis. It has an associated solution

$$x^*_{\tilde{B}} = A_{\tilde{B}}^{-1}b + A_{\tilde{B}}^{-1}A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynom with variable ε of degree at most m.

$A_{\tilde{B}}^{-1}A_B$ has rank m. Therefore no polynom is 0.

A polynom of degree at most m has at most m roots (Nullstellen).

Hence, $\varepsilon > 0$ small enough gives that no component of the above vector is 0. Hence, no degeneracies.

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- If it terminates because the reduced cost vector fulfills

$$\tilde{c} = (c^T - c_B^T A_B^{-1} A) \le 0$$

then we have found an optimal basis.

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- If it terminates because it finds a variable x_j with $\tilde{c}_j > 0$ for which the j-th basis direction d, fulfills $d \geq 0$ we know that LP' is unbounded. The basis direction does not depend on b. Hence, we also know that LP is unbounded.

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Doing calculations with perturbed instances may be costly. Also the right choice of ε is difficult.

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Simulate behaviour of LP' without explicitly doing a perturbation.

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Lexicographic Pivoting

We choose the entering variable arbitrarily as before ($\tilde{c}_e > 0$, of course).

If we do not have a choice for the leaving variable then LP' and LP do the same (i.e., choose the same variable).

Otherwise we have to be careful.

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Matrix View

Let our linear program be

$$\begin{array}{ccccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

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LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

$$\theta_\ell = \frac{\hat{b}_\ell}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_\ell}{(A_B^{-1}A_{*e})_\ell} \;.$$

ℓ is the index of a leaving variable within B. This means if e.g. $B = \{1, 3, 7, 14\}$ and leaving variable is 3 then $\ell = 2$.

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If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

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ℓ is the index of a leaving variable within B. This means if e.g. $B = \{1, 3, 7, 14\}$ and leaving variable is 3 then $\ell = 2$.

Matrix View

Let our linear program be

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Definition 44
$u \leq_{\text{lex}} v$ if and only if the first component in which u and v differ fulfills $u_i \leq v_i$.

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Example

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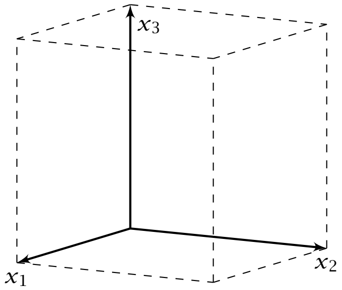

$2n$ constraint on n variables define an n-dimensional hypercube as feasible region.

The feasible region has 2^n vertices.

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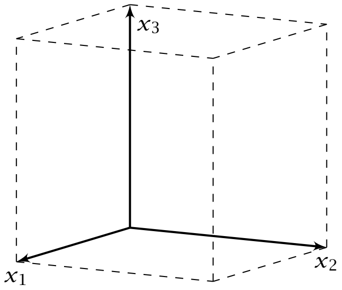

However, Simplex may still run quickly as it usually does not visit all feasible bases.

In the following we give an example of a feasible region for which there is a bad Pivoting Rule.

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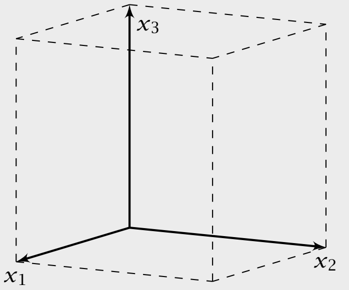

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Pivoting Rule

A Pivoting Rule defines how to choose the entering and leaving variable for an iteration of Simplex.

In the non-degenerate case after choosing the entering variable the leaving variable is unique.

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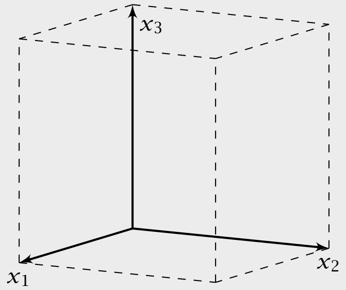

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Klee Minty Cube

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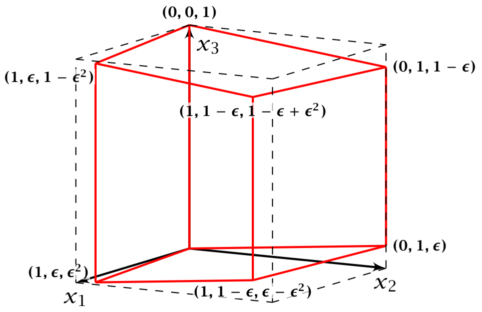

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- We have $2n$ constraints, and $3n$ variables (after adding slack variables to every constraint).
- Every basis is defined by $2n$ variables, and n non-basic variables.
- There exist degenerate vertices.
- The degeneracies come from the non-negativity constraints, which are superfluous.
- In the following all variables x_i stay in the basis at all times.
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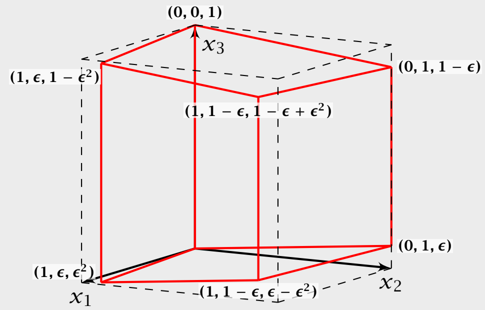

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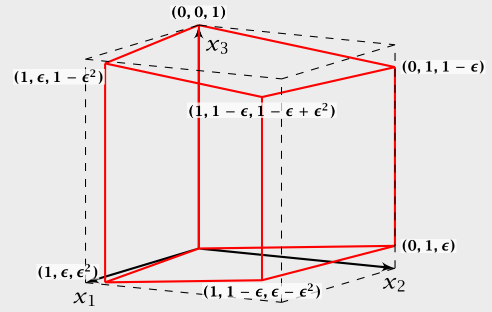

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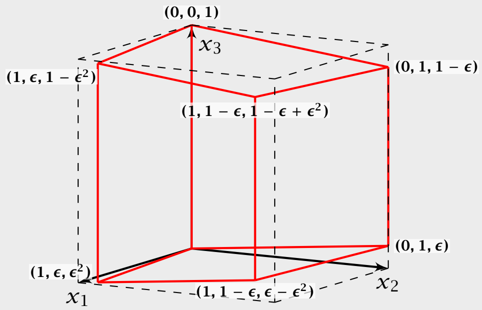

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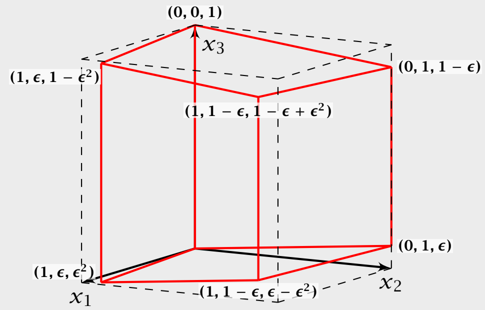

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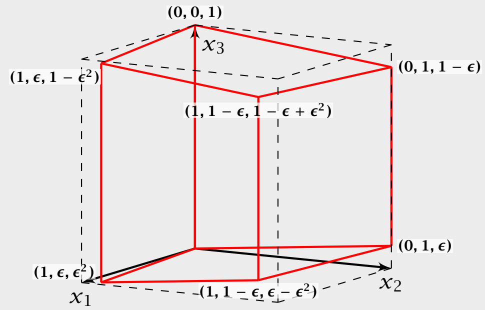

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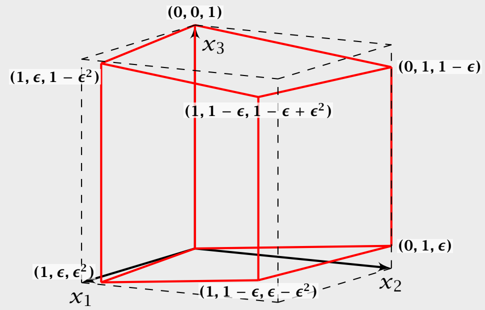

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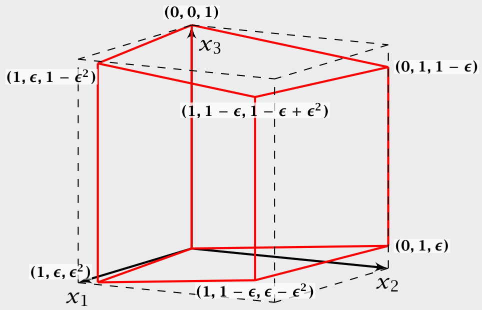

Analysis

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Observations

- We have $2n$ constraints, and $3n$ variables (after adding slack variables to every constraint).
- Every basis is defined by $2n$ variables, and n non-basic variables.
- There exist degenerate vertices.
- The degeneracies come from the non-negativity constraints, which are superfluous.
- In the following all variables x_i stay in the basis at all times.
- Then, we can uniquely specify a basis by choosing for each variable whether it should be equal to its lower bound, or equal to its upper bound (the slack variable corresponding to the non-tight constraint is part of the basis).
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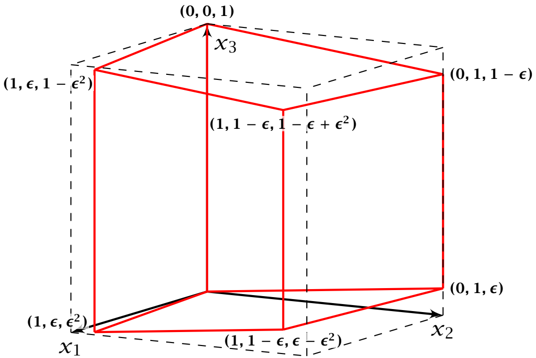

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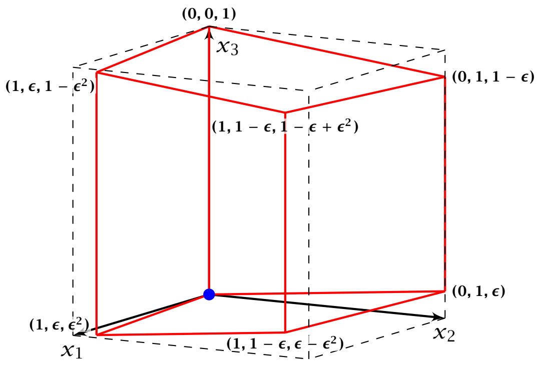

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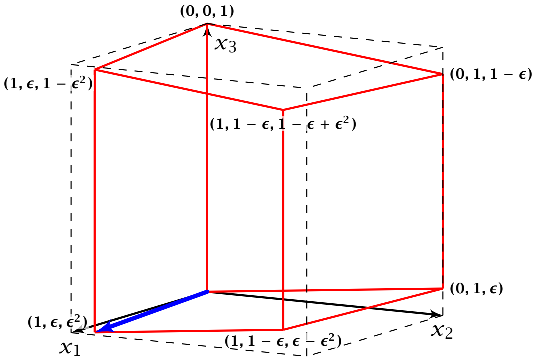

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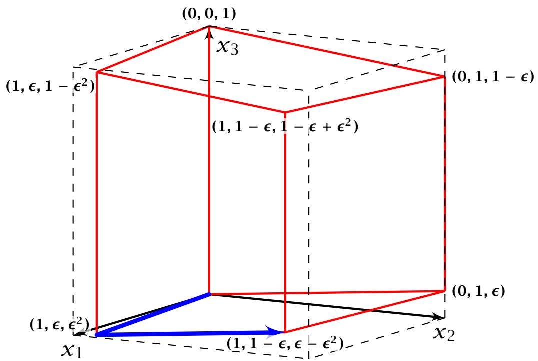

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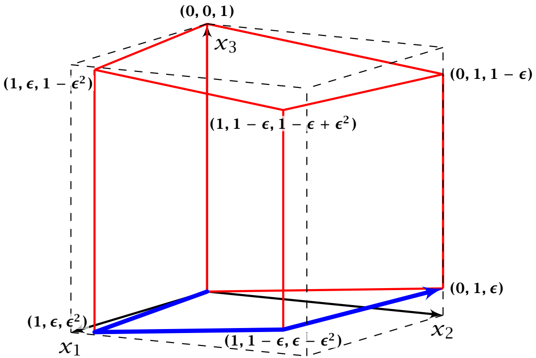

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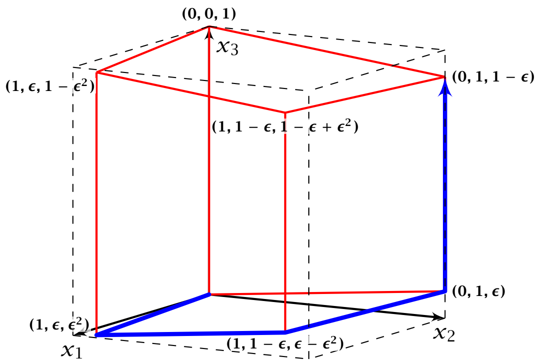

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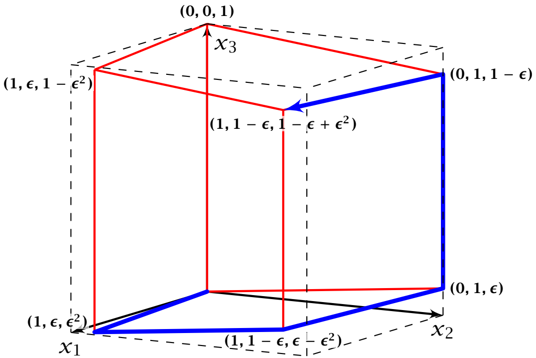

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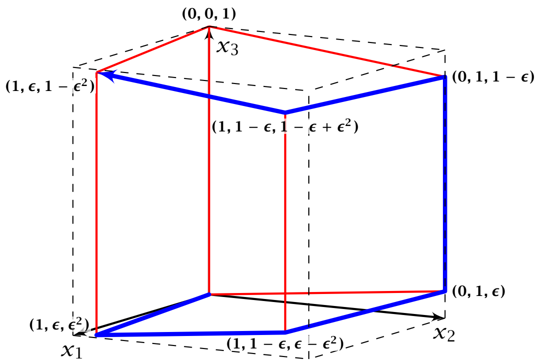

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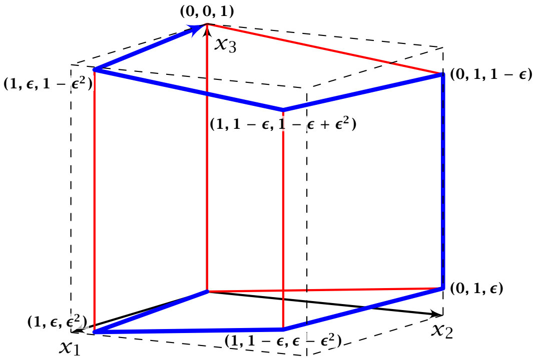

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Analysis

The sequence S_n that visits every node of the hypercube is defined recursively

$$
\begin{array}{cc}
(0,\dots,0,0,0) & \\
\big\downarrow S_{n-1} & \\
(0,\dots,0,1,0) & \\
\big\downarrow & S_n \\
(0,\dots,0,1,1) & \\
\big\downarrow S_{n-1}^{\text{rev}} & \\
(0,\dots,0,0,1) & \big\downarrow
\end{array}
$$

The non-recursive case is $S_1 = 0 \to 1$

Klee Minty Cube

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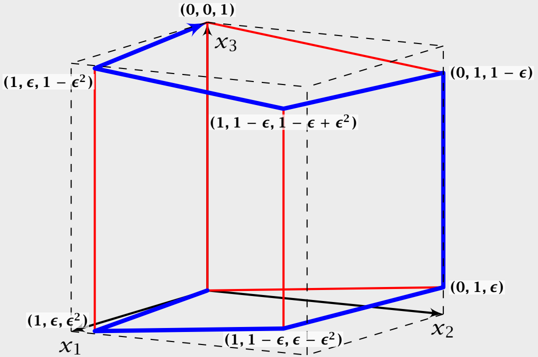

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Lemma 45
The objective value x_n is increasing along path S_n.

Proof by induction:

$n = 1$: obvious, since $S_1 = 0 \to 1$, and $1 > 0$.

$n - 1 \to n$

- For the first part the value of $x_n = \epsilon x_{n-1}$.
- By induction hypothesis x_{n-1} is increasing along S_{n-1}, hence, also x_n.
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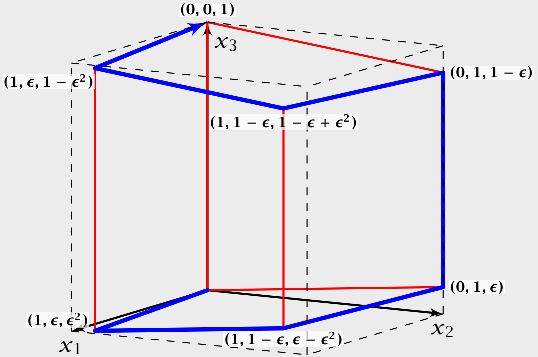

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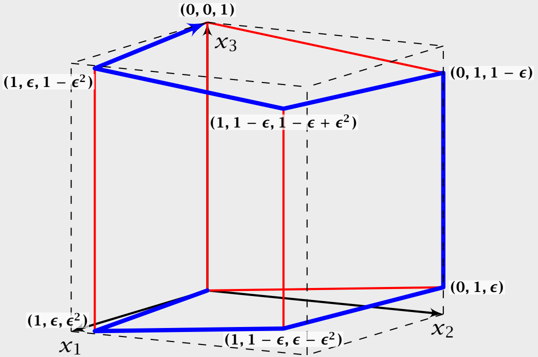

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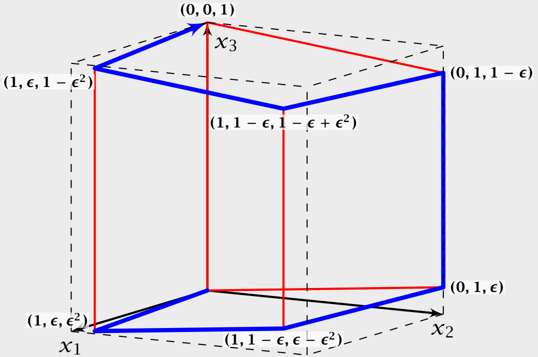

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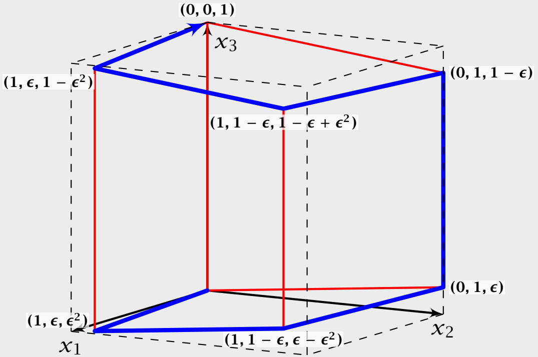

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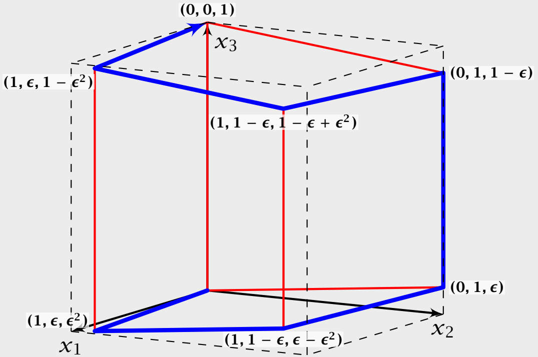

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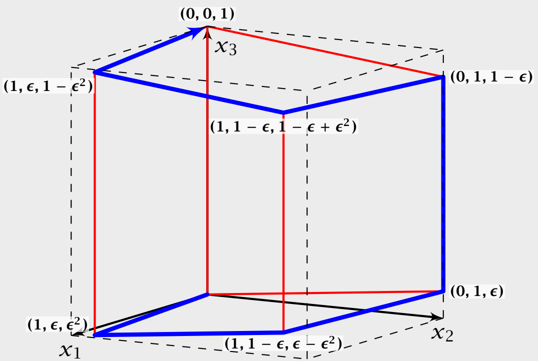

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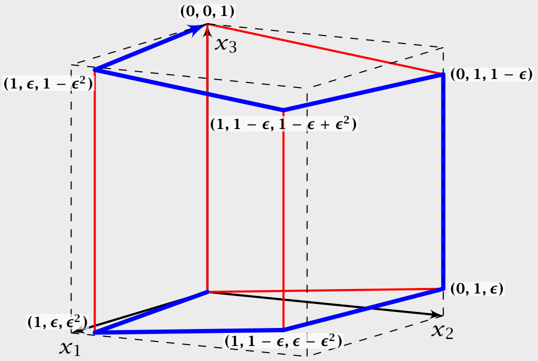

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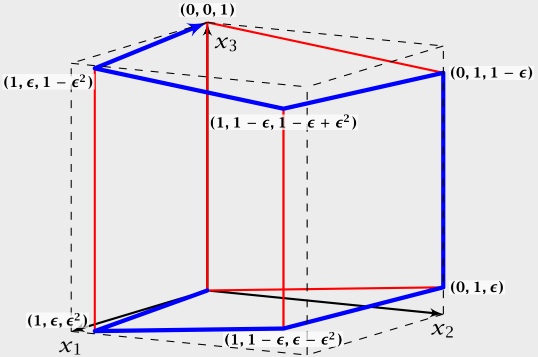

Remarks about Simplex

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The simplex algorithm takes at most $\binom{n}{m}$ iterations. Each iteration can be implemented in time $\mathcal{O}(mn)$.

In practise it usually takes a linear number of iterations.

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Remarks about Simplex

Theorem
For almost all known deterministic pivoting rules (rules for choosing entering and leaving variables) there exist lower bounds that require the algorithm to have exponential running time ($\Omega(2^{\Omega(n)})$) (e.g. Klee Minty 1972).

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For some standard randomized pivoting rules there exist subexponential lower bounds ($\Omega(2^{\Omega(n^\alpha)})$ for $\alpha > 0$) (Friedmann, Hansen, Zwick 2011).

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Theorem
For almost all known deterministic pivoting rules (rules for choosing entering and leaving variables) there exist lower bounds that require the algorithm to have exponential running time ($\Omega(2^{\Omega(n)})$) (e.g. Klee Minty 1972).

Remarks about Simplex

Conjecture (Hirsch 1957)
The edge-vertex graph of an m-facet polytope in d-dimensional Euclidean space has diameter no more than $m - d$.

The conjecture has been proven wrong in 2010.

But the question whether the diameter is perhaps of the form $\mathcal{O}(\text{poly}(m, d))$ is open.

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8 Seidels LP-algorithm

- Suppose we want to solve $\min\{c^Tx \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have m constraints.
- In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
- If d is much smaller than m one can do a lot better.
- In the following we develop an algorithm with running time $\mathcal{O}(d! \cdot m)$, i.e., linear in m.

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Ensuring Conditions

Given a standard minimization LP

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how can we obtain an LP of the required form?

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Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b.

Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

Add slack variables to A; denote the resulting matrix with $\bar{A}$.

If B is an optimal basis then x_B with $\bar{A}_B x_B = b$, gives an optimal assignment to the basis variables (non-basic variables are 0).

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Let M be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by

$$x_i = \frac{\det(M_j)}{\det(M)} ,$$

where M_i is the matrix obtained from M by replacing the i-th column by the vector b.

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Bounding the Determinant

Let Z be the maximum absolute entry occuring in $\bar{A}, \bar{b}$ or c. Let C denote the matrix obtained from $\bar{A}_B$ by replacing the j-th column with vector $\bar{b}$ (for some j).

Observe that

$$|\det(C)|$$

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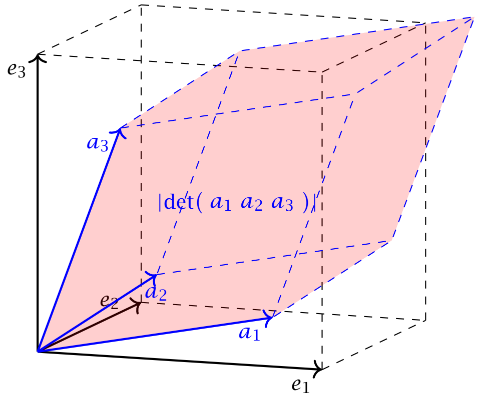

Hadamards inequality says that the volume of the red parallelepiped (Spat) is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

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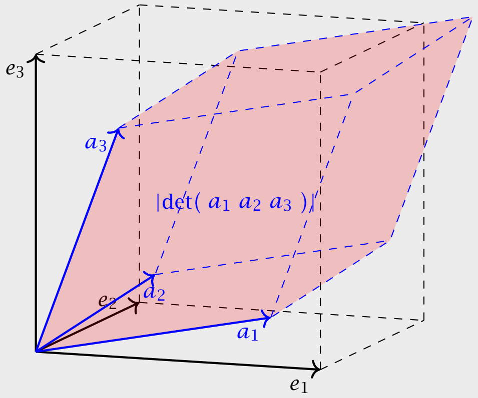

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Compute an optimum basis for the new LP.

- If the cost is $c^Tx = -(mZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
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In the following we use $\mathcal{H}$ to denote the set of all constraints apart from the constraint $c^Tx \geq -mZ(m! \cdot Z^m) - 1$.

We give a routine SeidelLP$(\mathcal{H}, d)$ that is given a set $\mathcal{H}$ of explicit, non-degenerate constraints over d variables, and minimizes c^Tx over all feasible points.

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- If we are unlucky and $\hat{x}^*$ does not fulfill h we need time $\mathcal{O}(d(m+1)) = \mathcal{O}(dm)$ to eliminate x_ℓ. Then we make a recursive call that takes time $T(m-1, d-1)$.
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This gives the recurrence

$$T(m,d) = \begin{cases} \mathcal{O}(\max\{1,m\}) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m-1,d) + & \\ \frac{d}{m}(\mathcal{O}(dm) + T(m-1,d-1)) & \text{otw.} \end{cases}$$

Note that $T(m,d)$ denotes the expected running time.

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Let C be the largest constant in the $\mathcal{O}$-notations.

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since $\sum_{i\geq 1}\frac{i^2}{i!}$ is a constant.

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Complexity

LP Feasibility Problem (LP feasibility)
Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}$ with $Ax = b$, $x \geq 0$?

The Bit Model

Input size

- The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- Let for an $m \times n$ matrix M, $L(M)$ denote the number of bits required to encode all the numbers in M.

$$\langle M \rangle := \sum_{i,j} \lceil \log_2(|m_{ij}|) + 1 \rceil$$

- In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
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- In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
- Then the input length is $L = \Theta(\langle A \rangle + \langle b \rangle)$.

Complexity

LP Feasibility Problem (LP feasibility)
Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}$ with $Ax = b$, $x \geq 0$?

- In the following we sometimes refer to $L := \langle A\rangle + \langle b\rangle$ as the input size (even though the real input size is something in $\Theta(\langle A\rangle + \langle b\rangle)$).
- In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in L).

The Bit Model

Input size

- The number of bits to represent a number $a \in \mathbb{Z}$ is

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- Let for an $m \times n$ matrix M, $L(M)$ denote the number of bits required to encode all the numbers in M.

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Suppose that $Ax = b; x \geq 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

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Size of a Basic Feasible Solution

Lemma 47

Let $M \in \mathbb{Z}^{m\times m}$ be an invertible matrix and let $b \in \mathbb{Z}^m$. Further define $L = \langle M\rangle + \langle b\rangle + n\log_2 n$. Then a solution to $Mx = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \le 2^L$ and $|D| \le 2^L$.

Proof:

Cramers rules says that we can compute x_j as

$$x_j = \frac{\det(M_j)}{\det(M)}$$

where M_j is the matrix obtained from M by replacing the j-th column by the vector b.

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EADS II
Harald Räcke
9 The Ellipsoid Algorithm

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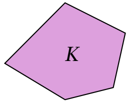

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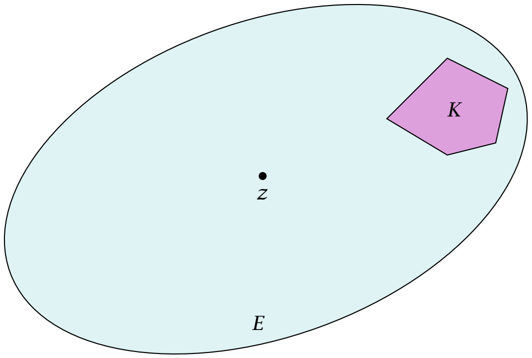

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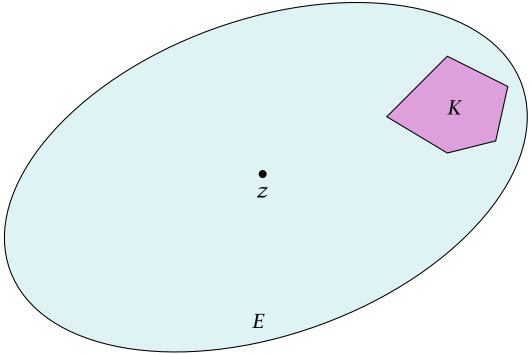

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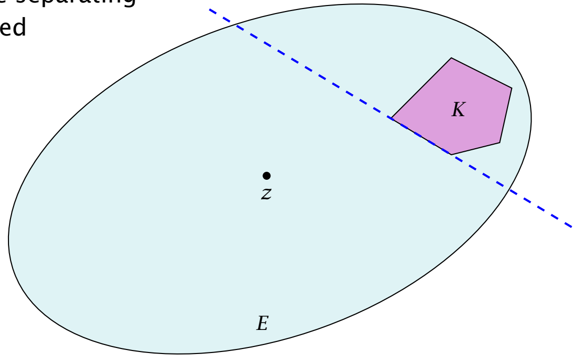

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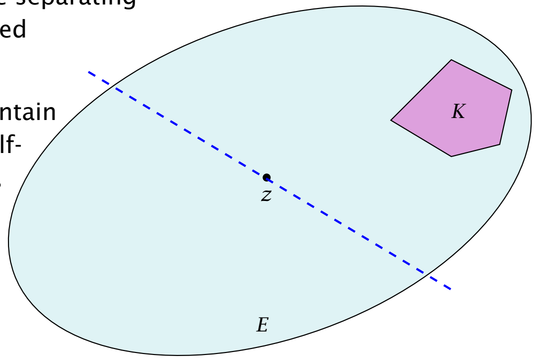

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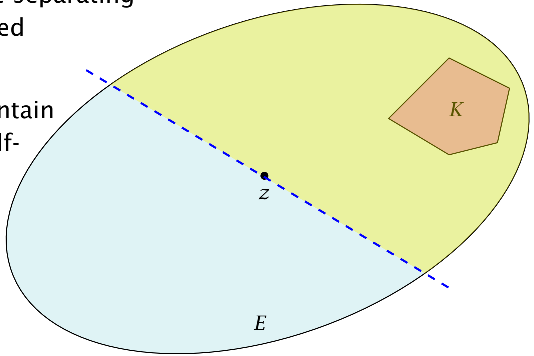

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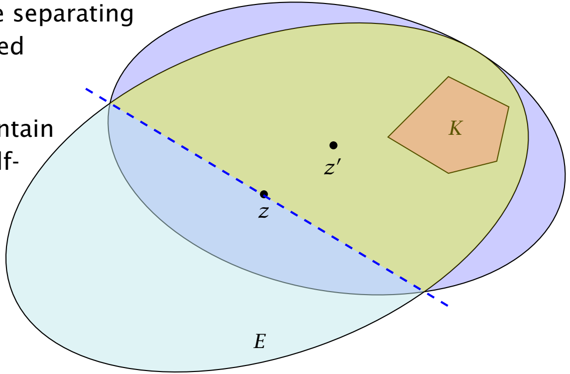

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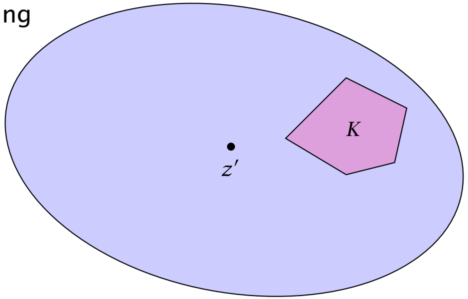

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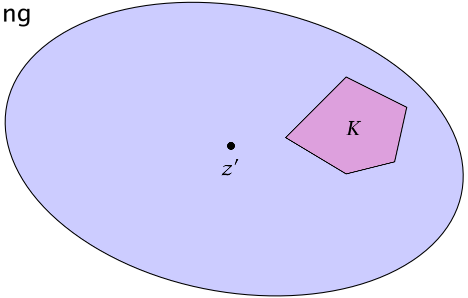
- Compute (smallest) ellipsoid E' that contains $E \cap H$.
- REPEAT



How do we detect whether the LP is unbounded?

Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

We can add a constraint $c^T x \geq M_{\max} + 1$ and check for feasibility.

Issues/Questions:

- How do you choose the first Ellipsoid? What is its volume?
- How do you measure progress? By how much does the volume decrease in each iteration?
- When can you stop? What is the minimum volume of a non-empty polytop?

Ellipsoid Method

- Let K be a convex set.
- Maintain ellipsoid E that is guaranteed to contain K provided that K is non-empty.
- If center $z \in K$ STOP.
- Otw. find a hyperplane separating K from z (e.g. a violated constraint in the LP).
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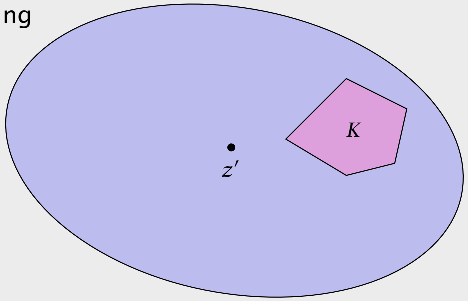

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A ball in $\mathbb{R}^n$ with center c and radius r is given by

$$\begin{aligned} B(c,r) &= \{x \mid (x-c)^T(x-c) \le r^2\} \\ &= \{x \mid \sum_i (x-c)_i^2/r^2 \le 1\} \end{aligned}$$

$B(0,1)$ is called the unit ball.

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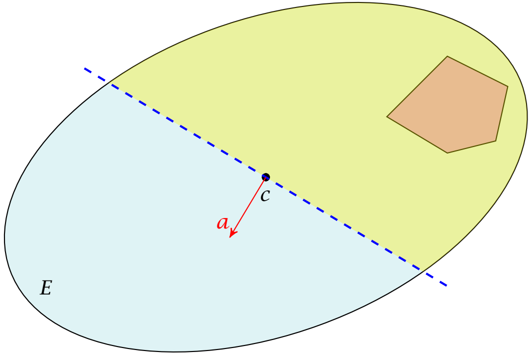
How to Compute the New Ellipsoid



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EADS II
Harald Räcke
9 The Ellipsoid Algorithm

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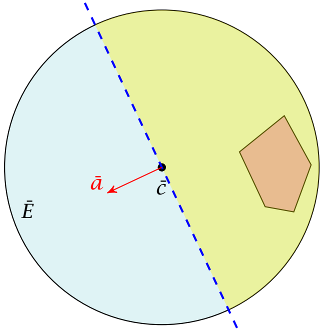

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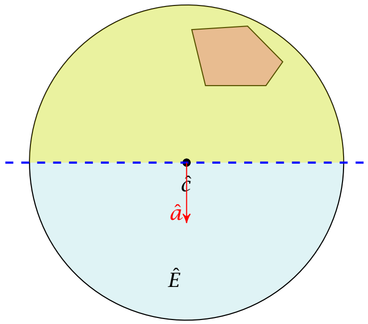

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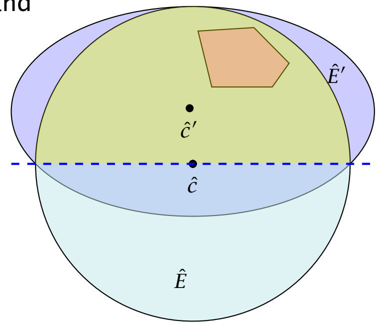

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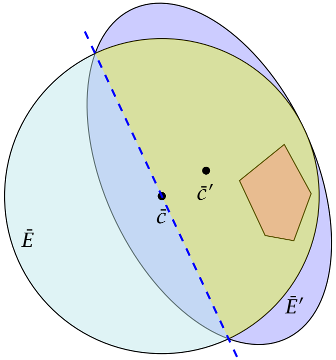

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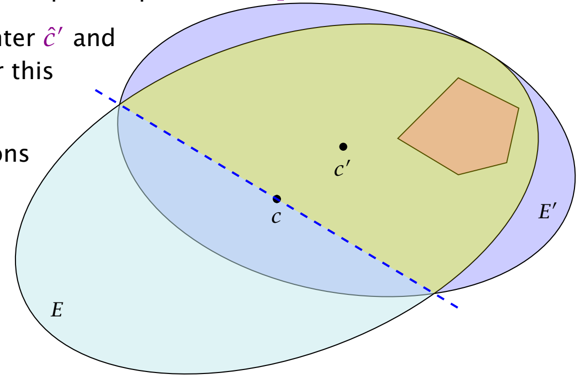

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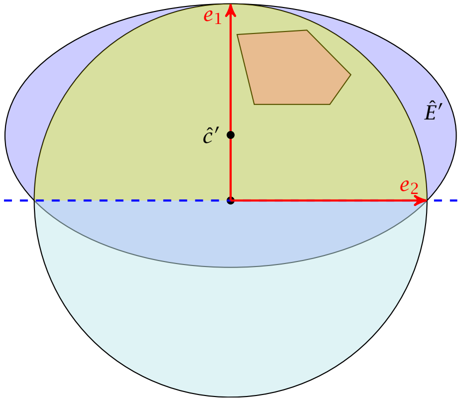

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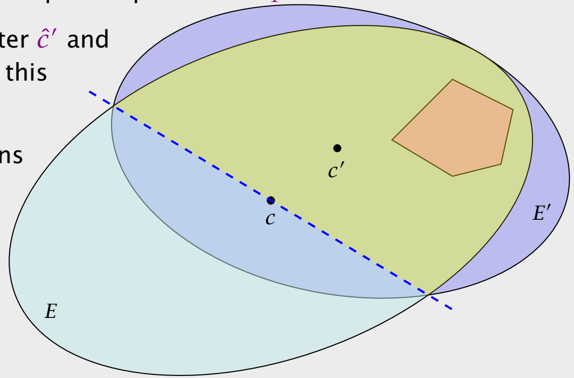

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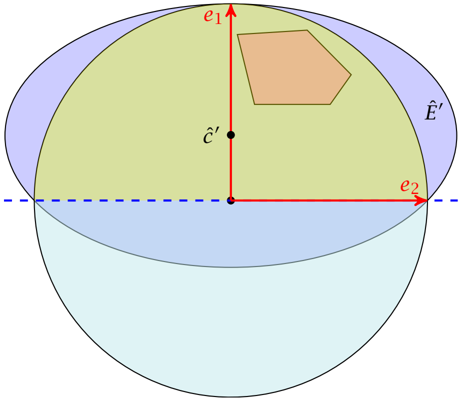

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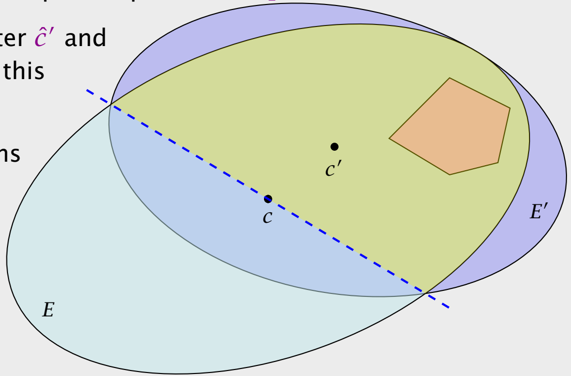

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- To obtain the matrix $\hat{Q}'^{-1}$ for our ellipsoid $\hat{E}'$ note that $\hat{E}'$ is axis-parallel.
- Let a denote the radius along the x_1-axis and let b denote the (common) radius for the other axes.
- The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

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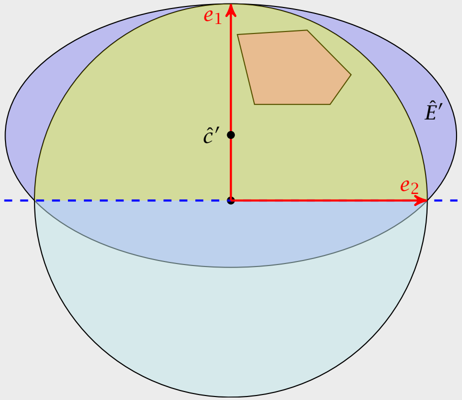

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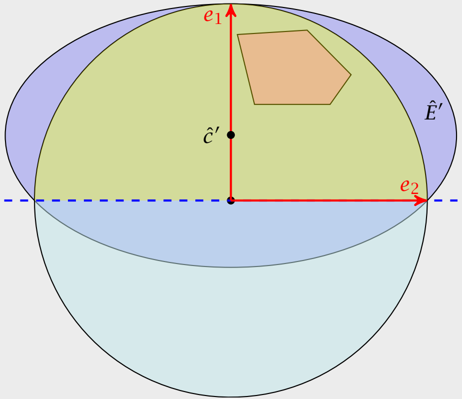

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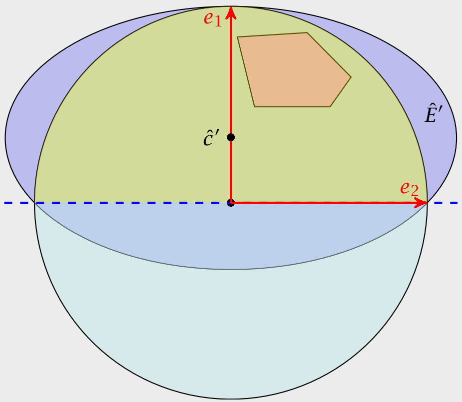

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$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

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$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

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- For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$ looks like (here $i = 2$)

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2}$$

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$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2} = 1 - \frac{t^2}{(1-t)^2}$$

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- This gives $(1-t)^2 = a^2$.

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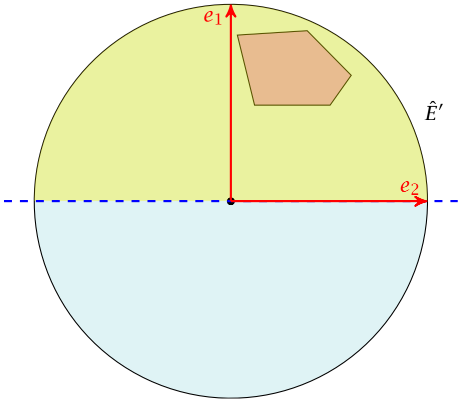
We still have many choices for t:



Choose t such that the volume of $\hat{E}'$ is minimal!!!

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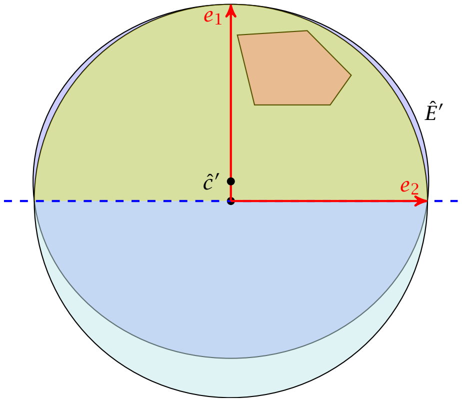

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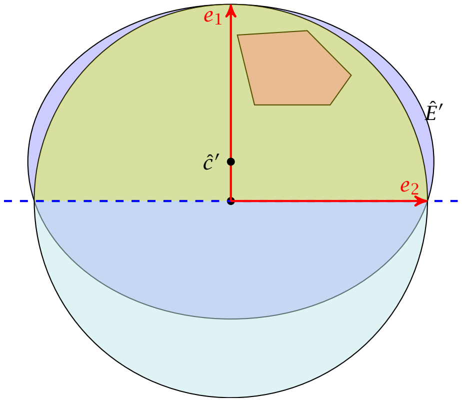

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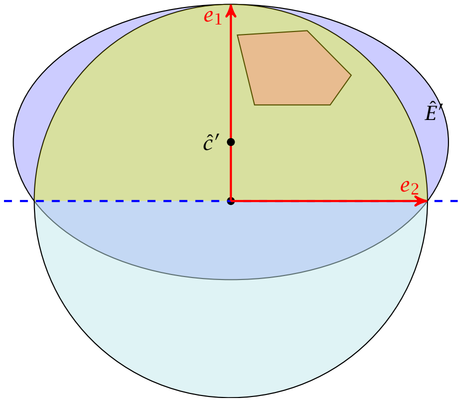

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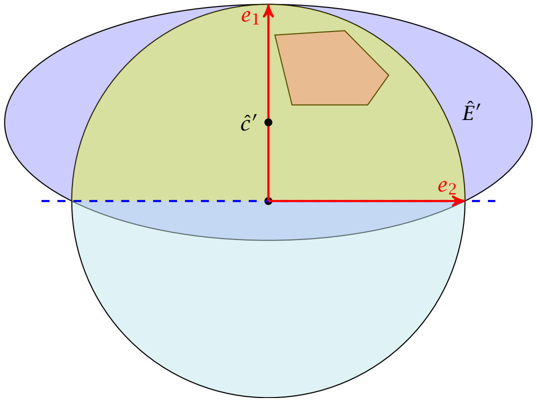

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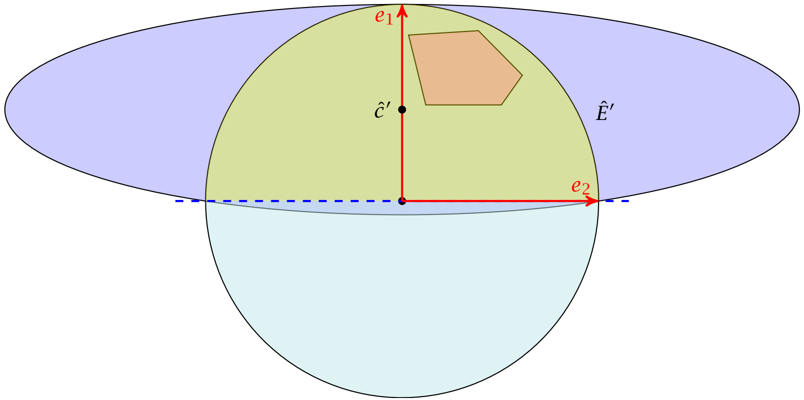

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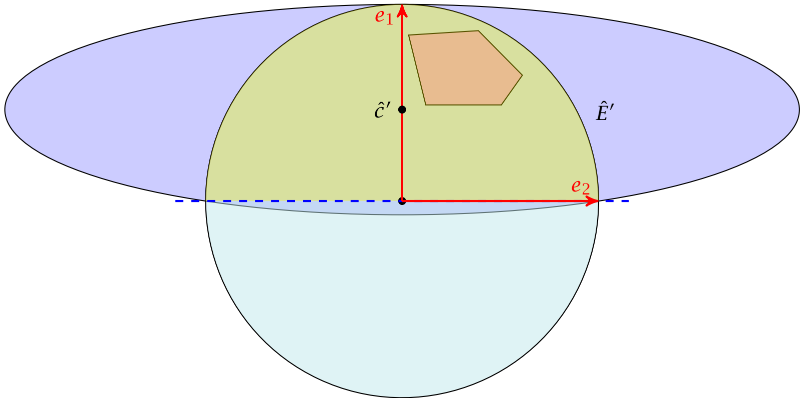

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We want to choose t such that the volume of $\hat{E}'$ is minimal.

Lemma 51

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\mathrm{vol}(L(K)) = |\det(L)| \cdot \mathrm{vol}(K) \ .$$

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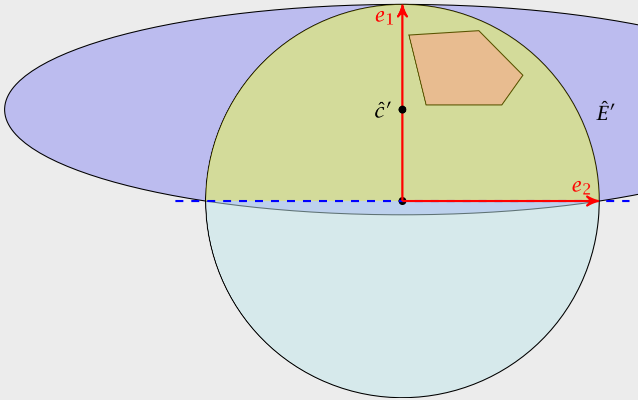

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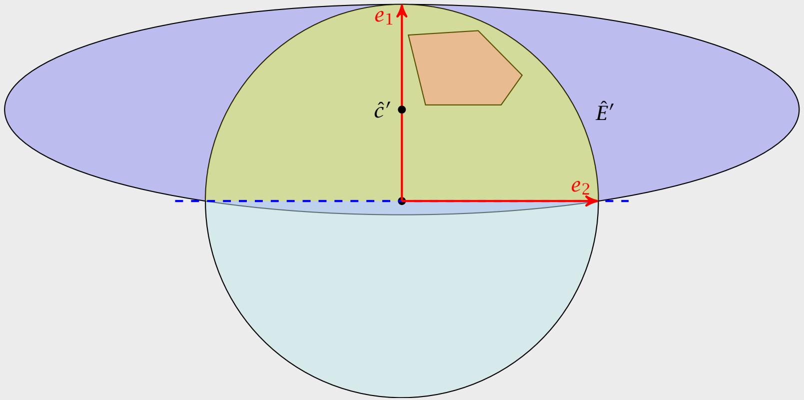

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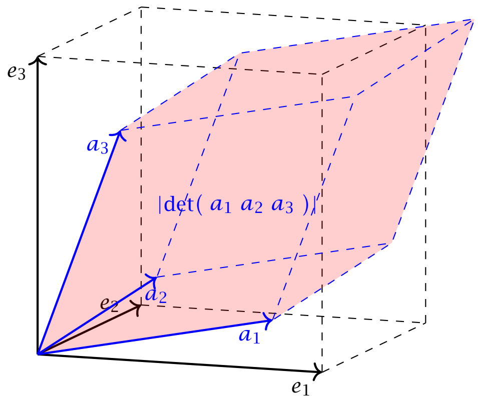
n-dimensional volume



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- We want to choose t such that the volume of $\hat{E}'$ is minimal.

$$\mathrm{vol}(\hat{E}') = \mathrm{vol}(B(0,1)) \cdot |\det(\hat{L}')| \enspace ,$$

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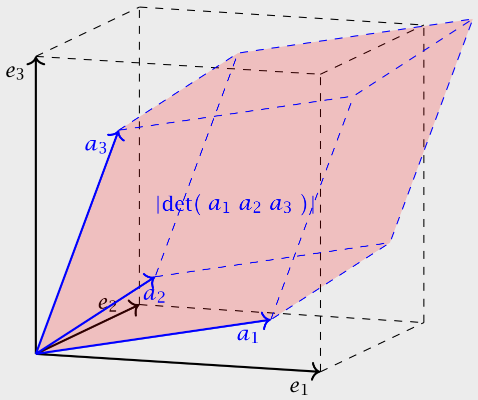

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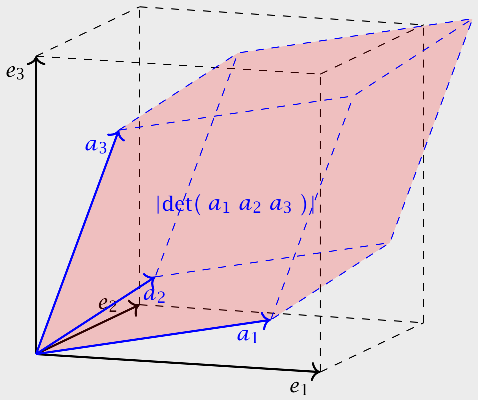

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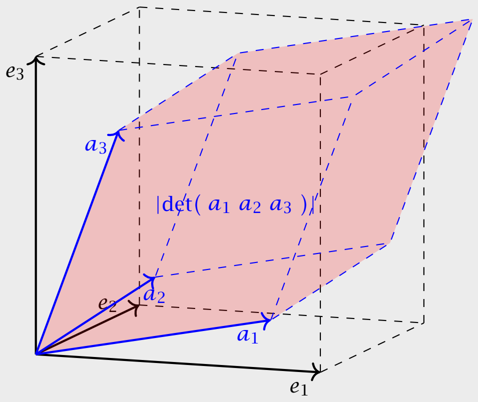

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$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

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$$= \frac{\Phi}{N^2}$$

$\boxed{N = \text{denominator}}$

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derivative of numerator

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$$-(n-1)(\sqrt{1-2t})^{n-2}$$

outer derivative

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$$-(n-1)(\sqrt{1-2t})^{n-2}\cdot\underbrace{\frac{1}{2\sqrt{1-2t}}\cdot(-2)}_{\text{inner derivative}}$$

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$$= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}$$

The Easy Case

$$\begin{aligned}\mathrm{vol}(\hat{E}') &= \mathrm{vol}(B(0,1))\cdot|\det(\hat{L}')| \\ &= \mathrm{vol}(B(0,1))\cdot ab^{n-1} \\ &= \mathrm{vol}(B(0,1))\cdot(1-t)\cdot\left(\frac{1-t}{\sqrt{1-2t}}\right)^{n-1} \\ &= \mathrm{vol}(B(0,1))\cdot\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\end{aligned}$$

We use the shortcut $\Phi := \mathrm{vol}(B(0,1))$.

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$$\begin{aligned}\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\ &= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n(1-t)^{n-1}\cdot \overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}}\\ &\quad -(n-1)\cancel{(\sqrt{1-2t})}^{n-2}\cdot\frac{1}{2\cancel{\sqrt{1-2t}}}\cdot(-2)\cdot(1-t)^n\Bigg)\\ &= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\end{aligned}$$

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&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}} \\
&\quad -(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{2\cancel{\sqrt{1-2t}}}\cdot(-2)\cdot(1-t)^n\Bigg) \\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}
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&\qquad \cancel{+}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)
\end{aligned}
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$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

$$= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}}$$

$$\cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg)$$

$$= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}$$

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The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

a

The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

$$= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overbrace{\cancel{(\sqrt{1-2t})^{n-1}}}^{1-2t}$$

$$\cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot(\cancel{-2})\cdot\overbrace{\cancel{(1-t)^n}}^{1-t}\Bigg)$$

$$= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}$$

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The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a = 1 - t$$

The Easy Case

$$\begin{aligned}\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\ &= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overbrace{\cancel{(\sqrt{1-2t})^{n-1}}}^{1-2t}\\ &\quad\cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overbrace{\cancel{(1-t)^n}}^{1-t}\Bigg)\\ &= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\ &\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\ &= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)\end{aligned}$$

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- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a = 1 - t = \frac{n}{n+1}$$

The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

$$= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overbrace{\cancel{(\sqrt{1-2t})^{n-1}}}^{1-2t}$$

$$\cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot(\cancel{-2})\cdot\overbrace{\cancel{(1-t)^n}}^{1-t}\Bigg)$$

$$= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}$$

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- We obtain the minimum for $t = \frac{1}{n+1}$.
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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad \cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}$$

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- We obtain the minimum for $t = \frac{1}{n+1}$.
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$$a = 1 - t = \frac{n}{n+1} \text{ and } b = \frac{1-t}{\sqrt{1-2t}}$$

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\begin{aligned}
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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1} \\
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To see the equation for b, observe that

$$b^2$$

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\end{aligned}$$

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&= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \\
&\qquad \cdot \Big((n-1)(1-t) - n(1-2t) \Big) \\
&= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \cdot \Big((n+1)t - 1 \Big)
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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1} \\
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The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

$$= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}^{\,1-2t}\cdot\cancel{(\sqrt{1-2t})^{n-1}}$$

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The Easy Case

Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

$$\gamma_n^2$$

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$$\gamma_n^2 = \Big(\frac{n}{n+1}\Big)^2\Big(\frac{n^2}{n^2-1}\Big)^{n-1}$$

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Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

$$\begin{aligned}\gamma_n^2 &= \left(\frac{n}{n+1}\right)^2\left(\frac{n^2}{n^2-1}\right)^{n-1}\\ &= \left(1-\frac{1}{n+1}\right)^2\left(1+\frac{1}{(n-1)(n+1)}\right)^{n-1}\end{aligned}$$

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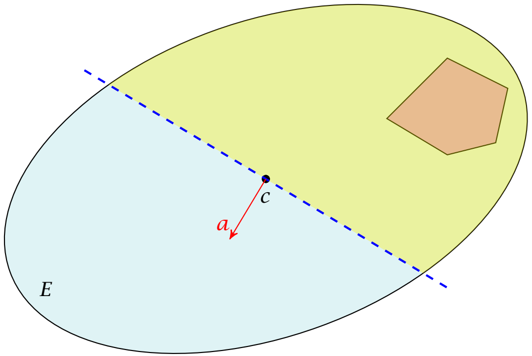
How to Compute the New Ellipsoid



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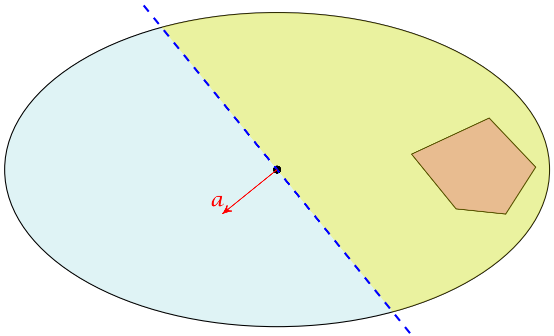

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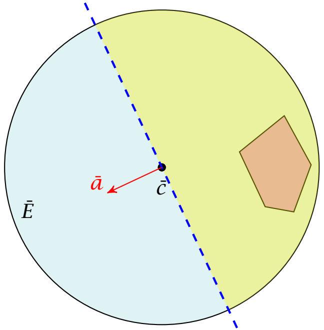

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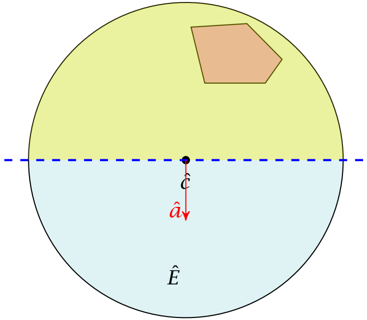

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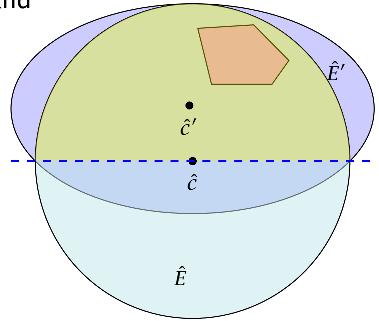

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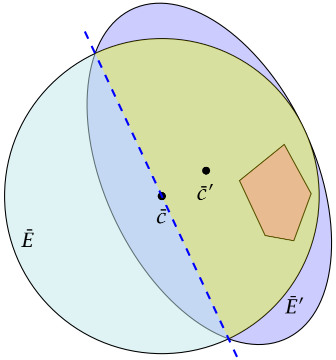

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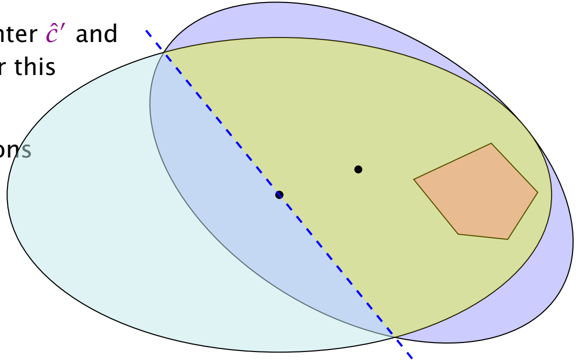

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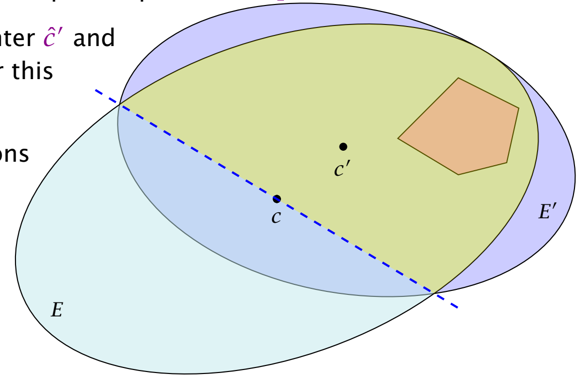

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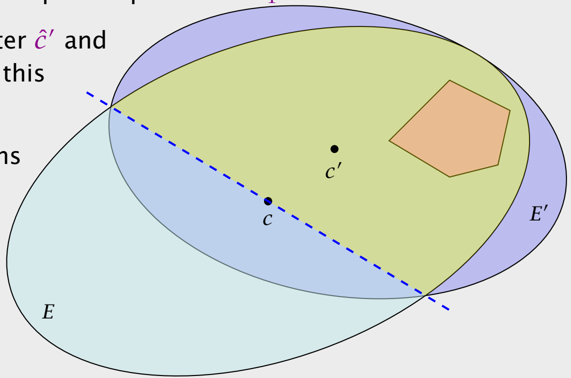

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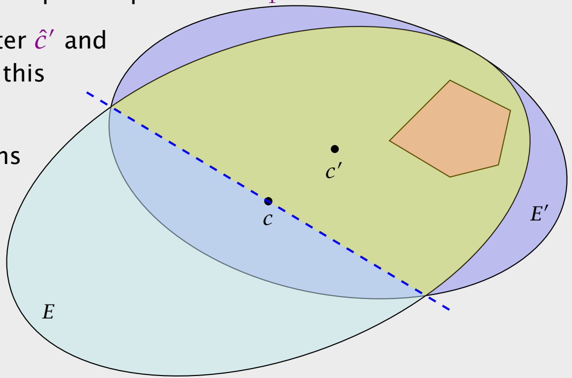

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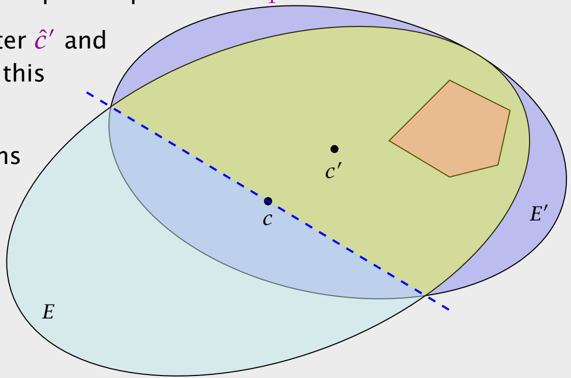

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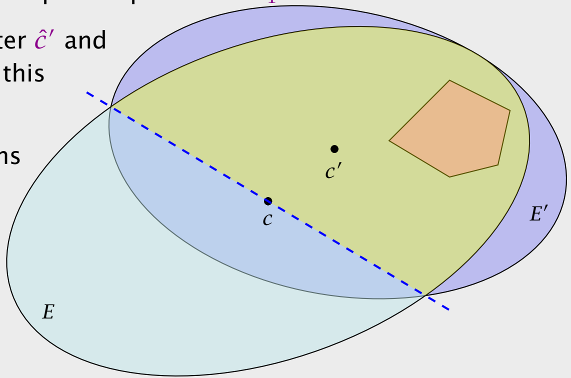

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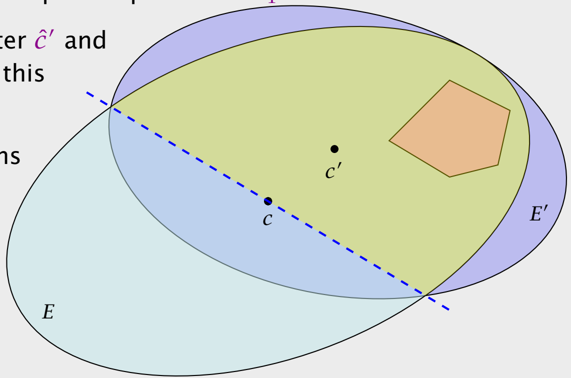

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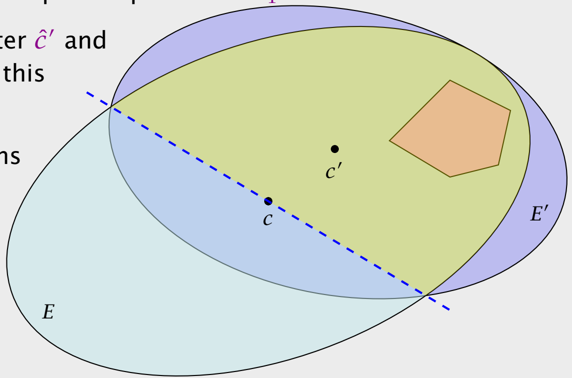

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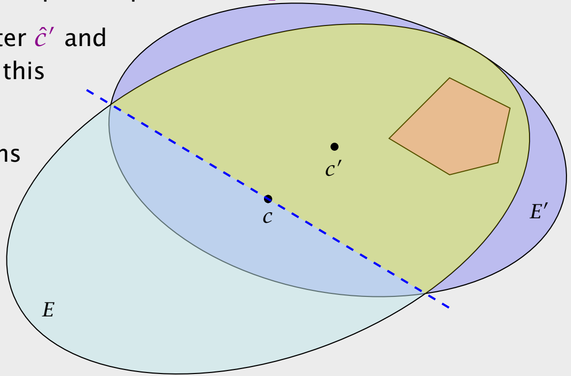

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Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

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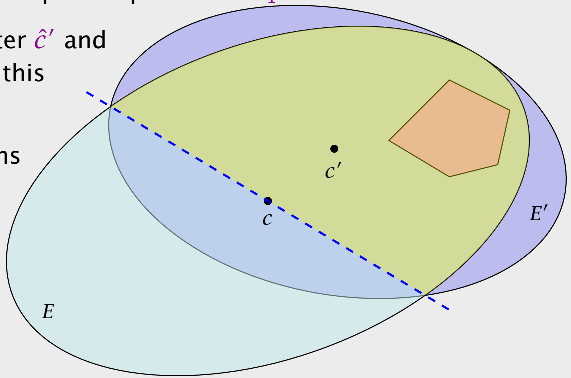

The Ellipsoid Algorithm

How to Compute The New Parameters?

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\begin{aligned}
e^{-\frac{1}{2(n+1)}} \geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} &= \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\
&= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}
\end{aligned}
$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

Our progress is the same:

$$e^{-\frac{1}{2(n+1)}} \geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))}$$
$$= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

Our progress is the same:

$$
\begin{aligned}
e^{-\frac{1}{2(n+1)}} \ge \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} &= \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\
&= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}
\end{aligned}
$$

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The Ellipsoid Algorithm

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The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \leq 0\}$;

$$f^{-1}(H) = \{f^{-1}(x) \mid a^T(x - c) \leq 0\}$$

Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} &\geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\ &= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

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The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

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$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y) - c) \le 0\} \end{aligned}$$

Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} &\ge \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\ &= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

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The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \end{aligned}$$

Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} \ge \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} &= \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\ &= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \leq 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \leq 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(Ly+c-c) \leq 0\} \end{aligned}$$

Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} \geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\ = \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

$$
\begin{aligned}
f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x - c) \le 0\} \\
&= \{f^{-1}(f(y)) \mid a^T(f(y) - c) \le 0\} \\
&= \{y \mid a^T(f(y) - c) \le 0\} \\
&= \{y \mid a^T(Ly + c - c) \le 0\} \\
&= \{y \mid (a^T L)y \le 0\}
\end{aligned}
$$

Our progress is the same:

$$
\begin{aligned}
e^{-\frac{1}{2(n+1)}} \ge \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} &= \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\
&= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}
\end{aligned}
$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^T L)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^T a$.

Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} \ge \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} &= \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\ &= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^T a}{\|L^T a\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \leq 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \leq 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(Ly + c - c) \leq 0\} \\ &= \{y \mid (a^T L)y \leq 0\} \end{aligned}$$

This means $\bar{a} = L^T a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^Ta}{\|L^Ta\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R\cdot e_1$$

Hence,

$$\bar{c}'$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \leq 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \leq 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(Ly+c-c) \leq 0\} \\ &= \{y \mid (a^TL)y \leq 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}'$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x - c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y) - c) \le 0\} \\ &= \{y \mid a^T(f(y) - c) \le 0\} \\ &= \{y \mid a^T(Ly + c - c) \le 0\} \\ &= \{y \mid (a^T L)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^T a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^TL)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^Ta}{\|L^Ta\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^Ta}{\|L^Ta\|}$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^TL)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^Ta}{\|L^Ta\|}$$

$$c'$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^TL)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^Ta}{\|L^Ta\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1}\frac{L^Ta}{\|L^Ta\|}$$

$$c' = f(\bar{c}')$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \leq 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \leq 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(Ly+c-c) \leq 0\} \\ &= \{y \mid (a^TL)y \leq 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

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After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1}\frac{L^Ta}{\|L^Ta\|}$$

$$c' = f(\bar{c}') = L \cdot \bar{c}' + c$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \leq 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \leq 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(f(y)-c) \leq 0\} \\ &= \{y \mid a^T(Ly+c-c) \leq 0\} \\ &= \{y \mid (a^TL)y \leq 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad\Rightarrow\quad -\frac{L^Ta}{\|L^Ta\|} = R\cdot e_1$$

Hence,

$$\bar{c}' = R\cdot\hat{c}' = R\cdot\frac{1}{n+1}e_1 = -\frac{1}{n+1}\frac{L^Ta}{\|L^Ta\|}$$

$$\begin{aligned} c' &= f(\bar{c}') = L\cdot\bar{c}' + c \\ &= -\frac{1}{n+1}L\frac{L^Ta}{\|L^Ta\|} + c \end{aligned}$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^TL)y \le 0\} \end{aligned}$$

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The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad\Rightarrow\quad -\frac{L^Ta}{\|L^Ta\|} = R\cdot e_1$$

Hence,

$$\bar{c}' = R\cdot\hat{c}' = R\cdot\frac{1}{n+1}e_1 = -\frac{1}{n+1}\frac{L^Ta}{\|L^Ta\|}$$

$$\begin{aligned} c' &= f(\bar{c}') = L\cdot\bar{c}' + c \\ &= -\frac{1}{n+1}L\frac{L^Ta}{\|L^Ta\|} + c \\ &= c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}} \end{aligned}$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^TL)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1}\frac{L^Ta}{\|L^Ta\|}$$

$$\begin{aligned} c' &= f(\bar{c}') = L \cdot \bar{c}' + c \\ &= -\frac{1}{n+1} L \frac{L^Ta}{\|L^Ta\|} + c \\ &= c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}} \end{aligned}$$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)$$

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2-1$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

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$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)$$

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For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)$$

because for $a^2 = {}^{n^2}/_{(n+1)^2}$ and $b^2 = {}^{n^2}/_{n^2-1}$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big)$$

because for $a^2 = {}^{n^2}/_{(n+1)^2}$ and $b^2 = {}^{n^2}/_{n^2-1}$

$$\begin{aligned} b^2 - b^2\frac{2}{n+1} &= \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2} \\ &= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2 \end{aligned}$$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)$$

because for $a^2 = {}^{n^2}/_{(n+1)^2}$ and $b^2 = {}^{n^2}/_{n^2-1}$

$$b^2 - b^2\frac{2}{n+1} = \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2}$$

$$= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2$$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)$$

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2-1$

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For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

9 The Ellipsoid Algorithm

$\bar{E}'$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

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$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1 e_1^T\Big)$$

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9 The Ellipsoid Algorithm

$$\bar{E}' = R(\hat{E}')$$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

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9 The Ellipsoid Algorithm

$$
\begin{aligned}
\bar{E}' &= R(\hat{E}') \\
&= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\}
\end{aligned}
$$

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9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

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&= \{y \mid y^T (\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1} y \le 1\}
\end{aligned}
$$

Recall that

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\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}
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\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big)
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9 The Ellipsoid Algorithm

Hence,

$\bar{Q}'$

9 The Ellipsoid Algorithm

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\begin{aligned}
\bar{E}' &= R(\hat{E}') \\
&= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\} \\
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&= \{y \mid y^T (R^T)^{-1} \hat{Q}'^{-1} R^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1} y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$\bar{Q}' = R\hat{Q}'R^T$$

9 The Ellipsoid Algorithm

$$\begin{aligned}\bar{E}' &= R(\hat{E}')\\ &= \{R(x) \mid x^T\hat{Q}'^{-1}x \le 1\}\\ &= \{y \mid (R^{-1}y)^T\hat{Q}'^{-1}R^{-1}y \le 1\}\\ &= \{y \mid y^T(R^T)^{-1}\hat{Q}'^{-1}R^{-1}y \le 1\}\\ &= \{y \mid y^T(\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1}y \le 1\}\end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$
\begin{aligned}
\bar{Q}' &= R\hat{Q}'R^T \\
&= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big) \cdot R^T
\end{aligned}
$$

9 The Ellipsoid Algorithm

$$
\begin{aligned}
\bar{E}' &= R(\hat{E}') \\
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&= \{y \mid y^T(\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1}y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}\bar{Q}' &= R\hat{Q}'R^T \\ &= R\cdot\frac{n^2}{n^2-1}\Big(I-\frac{2}{n+1}e_1e_1^T\Big)\cdot R^T \\ &= \frac{n^2}{n^2-1}\Big(R\cdot R^T-\frac{2}{n+1}(Re_1)(Re_1)^T\Big)\end{aligned}$$

9 The Ellipsoid Algorithm

$$\begin{aligned}\bar{E}' &= R(\hat{E}') \\ &= \{R(x)\mid x^T\hat{Q}'^{-1}x\le 1\} \\ &= \{y\mid (R^{-1}y)^T\hat{Q}'^{-1}R^{-1}y\le 1\} \\ &= \{y\mid y^T(R^T)^{-1}\hat{Q}'^{-1}R^{-1}y\le 1\} \\ &= \{y\mid y^T\underbrace{(R\hat{Q}'R^T)}_{\bar{Q}'}{}^{-1}y\le 1\}\end{aligned}$$

9 The Ellipsoid Algorithm

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&= \frac{n^2}{n^2-1}\Big(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big) \\
&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{\|L^Ta\|^2}\Big)
\end{aligned}
$$

9 The Ellipsoid Algorithm

$$
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&= \{y \mid y^T(R^T)^{-1}\hat{Q}'^{-1}R^{-1}y \le 1\} \\
&= \{y \mid y^T(\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1}y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

E'

9 The Ellipsoid Algorithm

Hence,

$$
\begin{aligned}
\bar{Q}' &= R\hat{Q}'R^T \\
&= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big) \cdot R^T \\
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\end{aligned}
$$

9 The Ellipsoid Algorithm

$$E' = L(\bar{E}')$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}
\bar{Q}' &= R\hat{Q}'R^T \\
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\end{aligned}$$

$$\begin{aligned} E' &= L(\bar{E}') \\ &= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

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9 The Ellipsoid Algorithm

$$
\begin{aligned}
E' &= L(\bar{E}') \\
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9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

Hence,

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9 The Ellipsoid Algorithm

Hence,

$$Q'$$

9 The Ellipsoid Algorithm

$$\begin{aligned}
E' &= L(\bar{E}') \\
&= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\
&= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{L\bar{Q}'L^T}_{Q'})^{-1} y \le 1\}
\end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$Q' = L\bar{Q}'L^T$$

9 The Ellipsoid Algorithm

$$\begin{aligned}
E' &= L(\bar{E}') \\
&= \{L(x) \mid x^T\bar{Q}'^{-1}x \le 1\} \\
&= \{y \mid (L^{-1}y)^T\bar{Q}'^{-1}L^{-1}y \le 1\} \\
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9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned} Q' &= L\bar{Q}'L^T \\ &= L \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{a^TQa}\Big) \cdot L^T \end{aligned}$$

9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

Hence,

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9 The Ellipsoid Algorithm

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Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
output: point $x \in K$ or "K is empty"
$Q \leftarrow$???
repeat
  if $c \in K$ **then return** c
  else
    choose a violated hyperplane a
    $$c \leftarrow c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}}$$
    $$Q \leftarrow \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Qaa^TQ}{a^TQa}\Big)$$
  endif
until ???
return "K is empty"

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned} Q' &= L\bar{Q}'L^T \\ &= L \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{a^TQa}\Big) \cdot L^T \\ &= \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Qaa^TQ}{a^TQa}\Big) \end{aligned}$$

Repeat: Size of basic solutions

Lemma 52

Let $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ *be a bounded polyhedron. Let* $\langle a_{\max}\rangle$ *be the maximum encoding length of an entry in* A, b*. Then every entry* x_j *in a basic solution fulfills* $|x_j| = \frac{D_j}{D}$ *with* $D_j, D \le 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

In the following we use $\delta := 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

Note that here we have $P = \{x \mid Ax \le b\}$. The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.

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$Q \leftarrow$???
repeat
  if $c \in K$ **then return** c
  else
    choose a violated hyperplane a
    $$c \leftarrow c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}}$$
    $$Q \leftarrow \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Qaa^TQ}{a^TQa}\Big)$$
  endif
until ???
return "K is empty"

Repeat: Size of basic solutions

Proof:
Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$, b, be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices $\bar{A}_B$ and $\bar{M}_j$ (matrix obt. when replacing the j-th column of $\bar{A}_B$ by b) can become at most

$$\begin{aligned} \det(\bar{A}_B), \det(\bar{M}_j) &\leq \|\vec{\ell}_{\max}\|^{2n} \\ &\leq (\sqrt{2n} \cdot 2^{\langle a_{\max}\rangle})^{2n} \leq 2^{2n\langle a_{\max}\rangle + 2n\log_2 n} , \end{aligned}$$

where $\vec{\ell}_{\max}$ is the longest column-vector that can be obtained after deleting all but $2n$ rows and columns from $\bar{A}$.

This holds because columns from I_m selected when going from $\bar{A}$ to $\bar{A}_B$ do not increase the determinant. Only the at most $2n$ columns from matrices A and $-A$ that $\bar{A}$ consists of contribute.

Repeat: Size of basic solutions

Lemma 52
Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a bounded polyhedron. Let $\langle a_{\max}\rangle$ be the maximum encoding length of an entry in A, b. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \leq 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

In the following we use $\delta := 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

Note that here we have $P = \{x \mid Ax \leq b\}$. The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.

How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \le \delta$.

Hence, P is contained in the cube $-\delta \le x_i \le \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n \operatorname{vol}(B(0,1)) \le (n\delta)^n \operatorname{vol}(B(0,1))$.

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When can we terminate?

Let $P := \{x \mid Ax \le b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop. Let $\langle a_{\max}\rangle$ be the encoding length of the largest entry in A or b.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \le b + \frac{1}{\lambda}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \right\} ,$$

where $\lambda = \delta^2 + 1$.

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Lemma 53

P_λ is feasible if and only if P is feasible.

$\Leftarrow$: obvious!

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$\Rightarrow$:

Consider the polyhedrons

$$\bar{P} = \left\{x \mid \begin{bmatrix} A & -A & I_m \end{bmatrix} x = b; x \geq 0\right\}$$

and

$$\bar{P}_\lambda = \left\{x \mid \begin{bmatrix} A & -A & I_m \end{bmatrix} x = b + \frac{1}{\lambda}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}; x \geq 0\right\} .$$

P is feasible if and only if $\bar{P}$ is feasible, and P_λ feasible if and only if $\bar{P}_\lambda$ feasible.

$\bar{P}_\lambda$ is bounded since P_λ and P are bounded.

$\Longrightarrow$:

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$\bar{P}_\lambda$ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1} b + \frac{1}{\lambda} \bar{A}_B^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x-values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

$$(\bar{A}_B^{-1} b)_i < 0 \le (\bar{A}_B^{-1} b)_i + \frac{1}{\lambda} (\bar{A}_B^{-1} \vec{1})_i$$

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$$(\bar{A}_B^{-1}b)_i < 0 \quad \Longrightarrow \quad (\bar{A}_B^{-1}b)_i \le -\frac{1}{\det(\bar{A}_B)}$$

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Lemma 54
If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \mathrm{vol}(B(0,1)) = \frac{1}{\delta^{3n}}\mathrm{vol}(B(0,1))$.

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Proof:

If P_λ feasible then also P. Let x be feasible for P.

By Cramers rule we get

$$(\bar{A}_B^{-1}b)_i < 0 \quad \Longrightarrow \quad (\bar{A}_B^{-1}b)_i \le -\frac{1}{\det(\bar{A}_B)}$$

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EADS II
Harald Räcke
9 The Ellipsoid Algorithm

How many iterations do we need until the volume becomes too small?

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input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
with $K \subseteq B(c,R)$, and $B(x,r) \subseteq K$ for some x
output: point $x \in K$ or "K is empty"
$Q \leftarrow \mathrm{diag}(R^2,\dots,R^2)$ // i.e., $L = \mathrm{diag}(R,\dots,R)$
repeat
  if $c \in K$ **then return** c
  else
    choose a violated hyperplane a
    $$c \leftarrow c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}}$$
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until $\det(Q) \le r^{2n}$ // i.e., $\det(L) \le r^n$
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Separation Oracle:

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- certifies that $x \in K$,
- or finds a hyperplane separating x from K.

We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- a guarantee that a ball of radius r is contained in K,
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The Ellipsoid algorithm requires $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

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    with $K \subseteq B(c, R)$, and $B(x, r) \subseteq K$ for some x
output: point $x \in K$ or "K is empty"
$Q \leftarrow \text{diag}(R^2, \dots, R^2)$ // i.e., $L = \text{diag}(R, \dots, R)$
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Separation Oracle:

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

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10 Karmarkars Algorithm

- inequalities $Ax \leq b$; $m \times n$ matrix A with rows a_i^T
- $P = \{x \mid Ax \leq b\}$; $P^\circ := \{x \mid Ax < b\}$
- interior point algorithm: $x \in P^\circ$ throughout the algorithm
- for $x \in P^\circ$ define
$$s_i(x) := b_i - a_i^T x$$
as the slack of the i-th constraint

logarithmic barrier function:

$$\phi(x) = -\sum_{i=1}^{m} \log(s_i(x))$$

Penalty for point x; points close to the boundary have a very large penalty.

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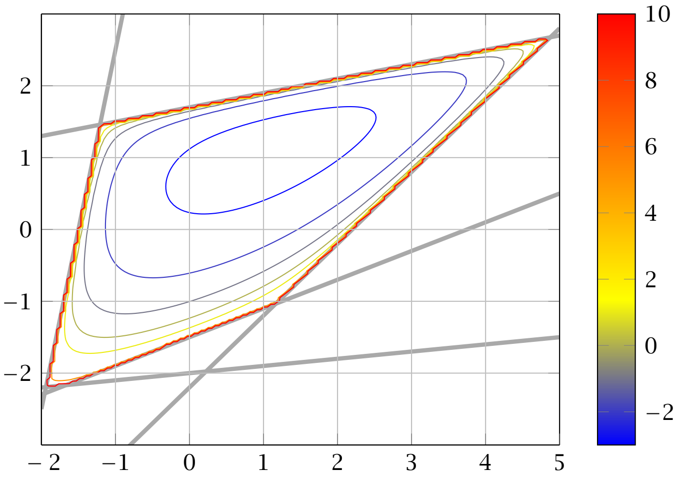

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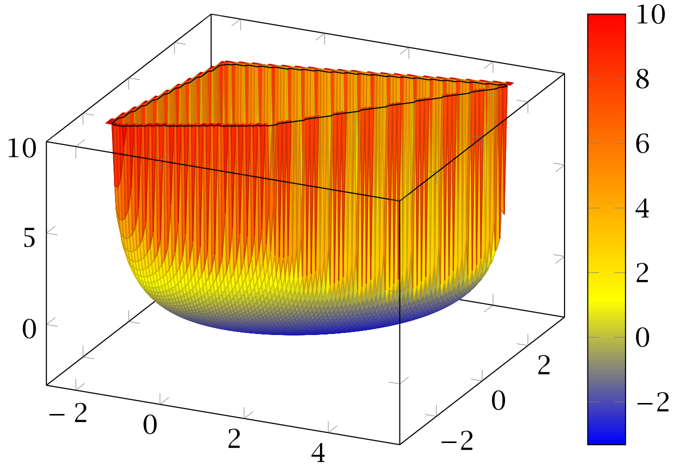

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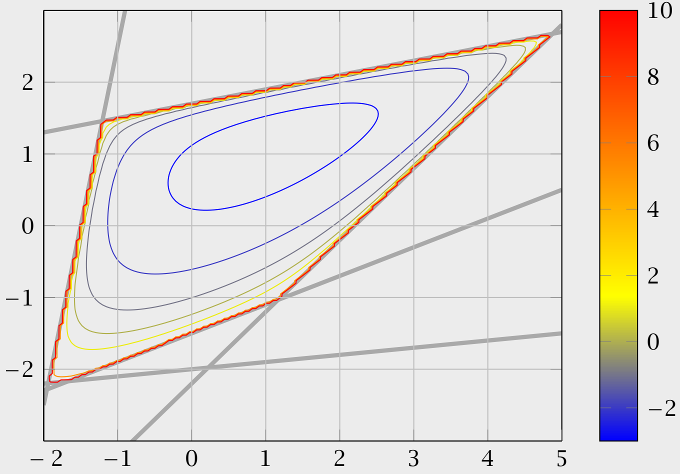

Gradient and Hessian

Taylor approximation:

$$\phi(x+\epsilon) \approx \phi(x) + \nabla\phi(x)^T\epsilon + \frac{1}{2}\epsilon^T\nabla^2\phi(x)\epsilon$$

Gradient:

$$\nabla\phi(x) = \sum_{i=1}^{m} \frac{1}{s_i(x)} \cdot a_i = A^T d_x$$

where $d_x^T = (1/s_1(x),\dots,1/s_m(x))$. ($d_x$ vector of inverse slacks)

Hessian:

$$H_x := \nabla^2\phi(x) = \sum_{i=1}^{m} \frac{1}{s_i(x)^2} a_i a_i^T = A^T D_x^2 A$$

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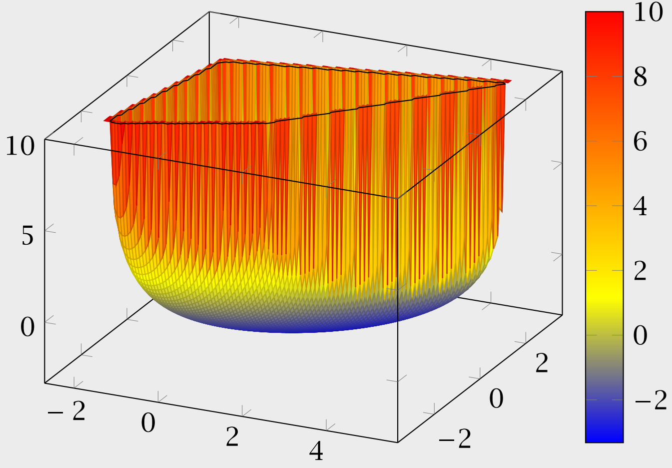

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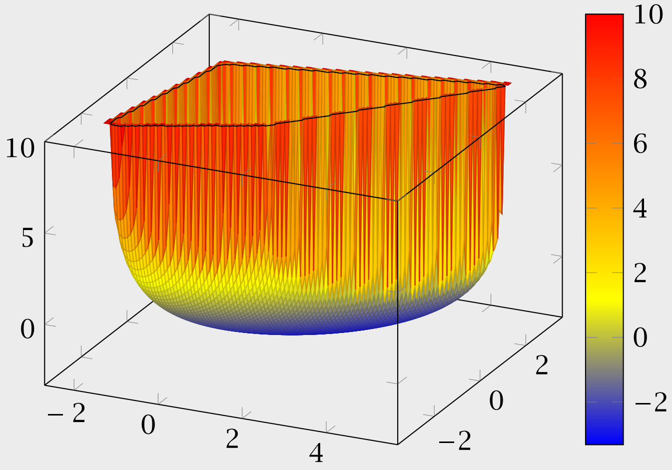

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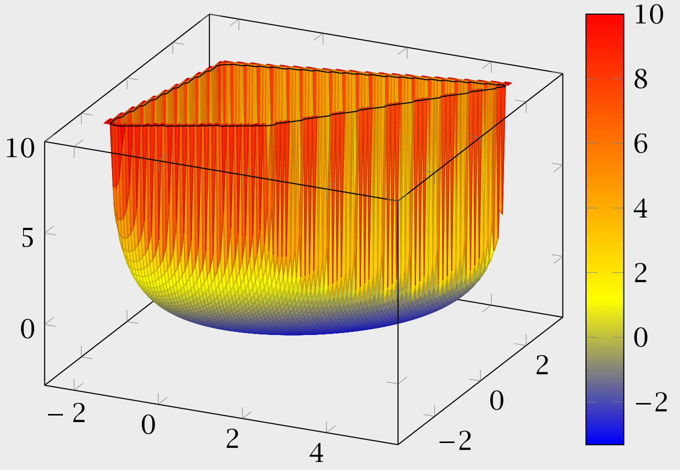

Proof for Gradient

$$
\begin{aligned}
\frac{\partial \phi(x)}{\partial x_i} &= \frac{\partial}{\partial x_i}\Big(-\sum_r \ln(s_r(x))\Big) \\
&= -\sum_r \frac{\partial}{\partial x_i}\Big(\ln(s_r(x))\Big) = -\sum_r \frac{1}{s_r(x)}\frac{\partial}{\partial x_i}\Big(s_r(x)\Big) \\
&= -\sum_r \frac{1}{s_r(x)}\frac{\partial}{\partial x_i}\Big(b_r - a_r^T x\Big) = \sum_r \frac{1}{s_r(x)}\frac{\partial}{\partial x_i}\Big(a_r^T x\Big) \\
&= \sum_r \frac{1}{s_r(x)} A_{ri}
\end{aligned}
$$

The i-th entry of the gradient vector is $\sum_r 1/s_r(x) \cdot A_{ri}$. This gives that the gradient is

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\nabla\phi(x) = \sum_r 1/s_r(x) a_r = A^T d_x
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\phi(x+\epsilon) \approx \phi(x) + \nabla\phi(x)^T\epsilon + \frac{1}{2}\epsilon^T\nabla^2\phi(x)\epsilon
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Proof for Hessian

$$\frac{\partial}{\partial x_j}\Big(\sum_r \frac{1}{s_r(x)}A_{ri}\Big) = \sum_r A_{ri}\Big(-\frac{1}{s_r(x)^2}\Big)\cdot\frac{\partial}{\partial x_j}\Big(s_r(x)\Big)$$
$$= \sum_r A_{ri}\frac{1}{s_r(x)^2}A_{rj}$$

Note that $\sum_r A_{ri}A_{rj} = (A^TA)_{ij}$. Adding the additional factors $1/s_r(x)^2$ can be done with a diagonal matrix.

Hence the Hessian is
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Properties of the Hessian

H_x is positive semi-definite for $x \in P^\circ$

$$u^T H_x u = u^T A^T D_x^2 A u = \|D_x A u\|_2^2 \geq 0$$

This gives that $\phi(x)$ is convex.

If $\operatorname{rank}(A) = n$, H_x is positive definite for $x \in P^\circ$

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$\|u\|_{H_x} := \sqrt{u^T H_x u}$ is a (semi-)norm; the unit ball w.r.t. this norm is an ellipsoid.

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Dikin Ellipsoid

$$E_x = \{y \mid (y-x)^T H_x (y-x) \le 1\} = \{y \mid \|y-x\|_{H_x} \le 1\}$$

Points in E_x are feasible!!!

In order to become infeasible when going from x to y one of the terms in the sum would need to be larger than 1.

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Properties of the Hessian

H_x is positive semi-definite for $x \in P^\circ$

$$u^T H_x u = u^T A^T D_x^2 A u = \|D_x A u\|_2^2 \ge 0$$

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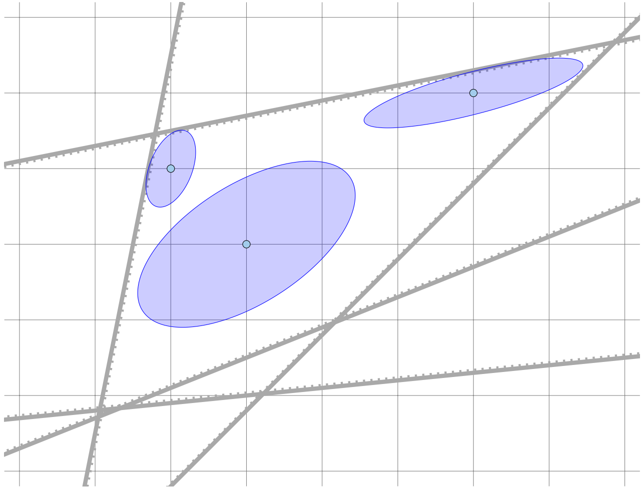

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Analytic Center

$$x_{\mathrm{ac}} := \arg\min_{x \in P^\circ} \phi(x)$$

- x_{ac} is solution to

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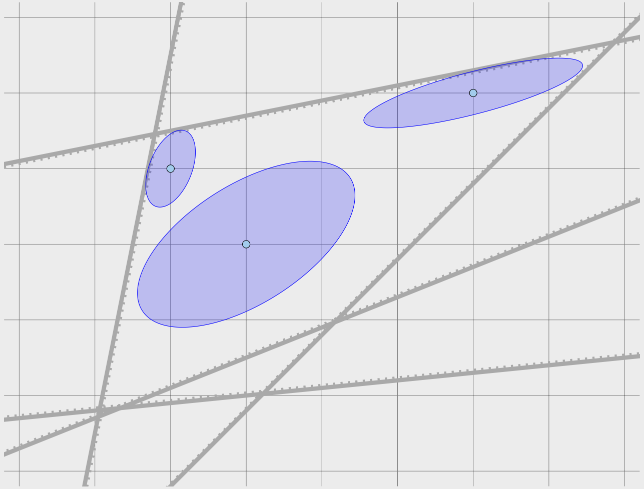

Central Path

In the following we assume that the LP and its dual are strictly feasible and that $\mathrm{rank}(A) = n$.

Central Path:
Set of points $\{x^*(t) \mid t > 0\}$ with

$$x^*(t) = \mathrm{argmin}_x \{tc^T x + \phi(x)\}$$

- $t = 0$: analytic center
- $t = \infty$: optimum solution

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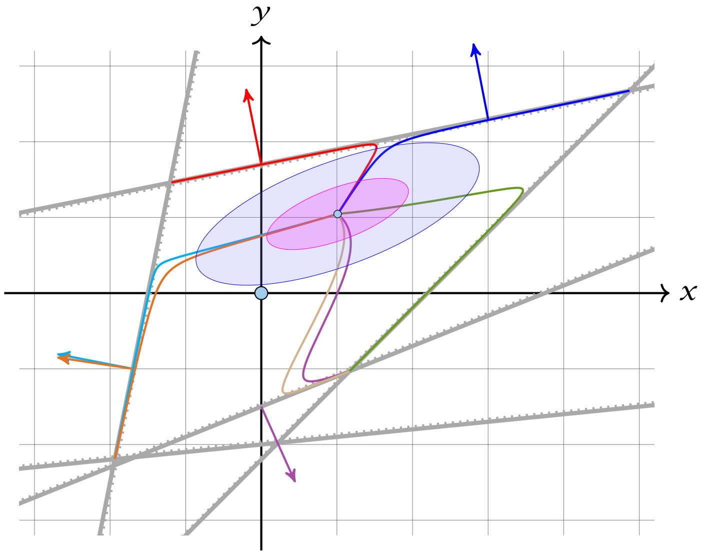

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Intuitive Idea:
Find point on central path for large value of t. Should be close to optimum solution.

Questions:

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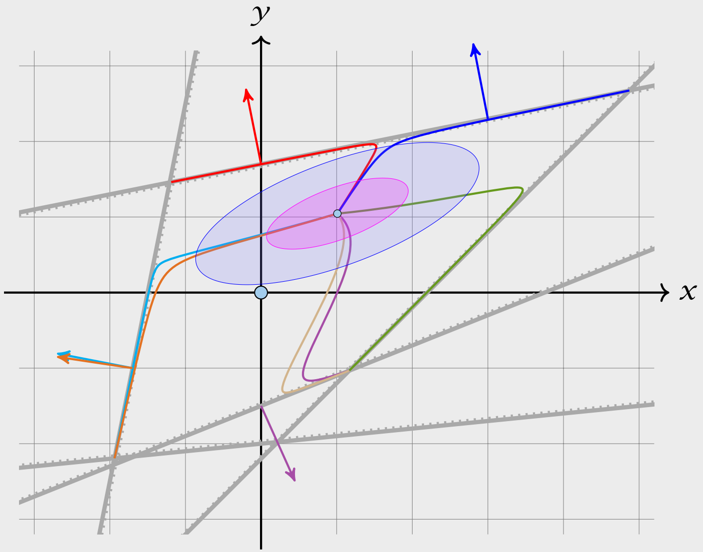

The Dual

primal-dual pair:

$$\begin{array}{ll} \text{min} & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

$$\begin{array}{ll} \text{max} & -b^T z \\ \text{s.t.} & A^T z + c = 0 \\ & z \geq 0 \end{array}$$

Assumptions

- primal and dual problems are strictly feasible;
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Force Field Interpretation

Point $x^*(t)$ on central path is solution to $tc + \nabla\phi(x) = 0$

- We can view each constraint as generating a repelling force. The combination of these forces is represented by $\nabla\phi(x)$.
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Point $x^*(t)$ on central path is solution to $tc + \nabla\phi(x) = 0$.

This means

$$tc + \sum_{i=1}^{m} \frac{1}{s_i(x^*(t))} a_i = 0$$

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$$c + \sum_{i=1}^{m} z_i^*(t) a_i = 0 \quad \text{with} \quad z_i^*(t) = \frac{1}{t s_i(x^*(t))}$$

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Newton Method

Quadratic approximation of f_t

$$f_t(x+\epsilon) \approx f_t(x) + \nabla f_t(x)^T\epsilon + \frac{1}{2}\epsilon^T H_{f_t}(x)\,\epsilon$$

Suppose this were exact:

$$f_t(x+\epsilon) = f_t(x) + \nabla f_t(x)^T\epsilon + \frac{1}{2}\epsilon^T H_{f_t}(x)\,\epsilon$$

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Newton decrement:

$$\begin{aligned} \lambda_t(x) &= \|D_x A \Delta x_{\text{nt}}\| \\ &= \|\Delta x_{\text{nt}}\|_{H_x} \end{aligned}$$

Square of Newton decrement is linear estimate of reduction if we do a Newton step:

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$$x := x + \Delta x_{\text{nt}}$$

Measuring Progress of Newton Step

Newton decrement:

$$\begin{aligned}\lambda_t(x) &= \|D_x A \Delta x_{\text{nt}}\| \\ &= \|\Delta x_{\text{nt}}\|_{H_x}\end{aligned}$$

Square of Newton decrement is linear estimate of reduction if we do a Newton step:

$$-\lambda_t(x)^2 = \nabla f_t(x)^T \Delta x_{\text{nt}}$$

- $\lambda_t(x) = 0$ iff $x = x^*(t)$
- $\lambda_t(x)$ is measure of proximity of x to $x^*(t)$

Newton Method

We want to move to a point where this gradient is 0:

Newton Step at $x \in P^\circ$

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Theorem 55

If $\lambda_t(x) < 1$ *then*

- $x_+ := x + \Delta x_{nt} \in P^\circ$ *(new point feasible)*
- $\lambda_t(x_+) \leq \lambda_t(x)^2$

This means we have quadratic convergence. Very fast.

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feasibility:

- $\lambda_t(x) = \|\Delta x_{\text{nt}}\|_{H_x} < 1$; hence x_+ lies in the Dikin ellipsoid around x.

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Convergence of Newtons Method

bound on $\lambda_t(x^+)$:
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To see the last equality we use Pythagoras

$$\|a\|^2 + \|a+b\|^2 = \|b\|^2$$

if $a^T(a+b) = 0$.

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$$\begin{aligned} DA\Delta x_{\mathrm{nt}} &= DA(x^+ - x) \\ &= D(b - Ax - (b - Ax^+)) \\ &= D(D^{-1}\vec{1} - D_+^{-1}\vec{1}) \\ &= (I - D_+^{-1}D)\vec{1} \end{aligned}$$

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Try to avoid this case!!!

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Path-following Methods

Try to slowly travel along the central path.

Algorithm 1 PathFollowing

1: start at analytic center
2: **while** solution not good enough **do**
3: make step to improve objective function
4: recenter to return to central path

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Short Step Barrier Method

simplifying assumptions:

- a first central point $x^*(t_0)$ is given
- $x^*(t)$ is computed exactly in each iteration

ϵ is approximation we are aiming for

start at $t = t_0$, repeat until $m/t \le \epsilon$

- compute $x^*(\mu t)$ using Newton starting from $x^*(t)$
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where $\mu = 1 + 1/(2\sqrt{m})$

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gradient of f_{t^+} at ($x = x^*(t)$)

$$\begin{aligned}\nabla f_{t^+}(x) &= \nabla f_t(x) + (\mu - 1)tc \\ &= -(\mu - 1)A^T D_x \vec{1}\end{aligned}$$

This holds because $0 = \nabla f_t(x) = tc + A^T D_x \vec{1}$.

The Newton decrement is

$$\begin{aligned}\lambda_{t^+}(x)^2 &= \nabla f_{t^+}(x)^T H^{-1} \nabla f_{t^+}(x) \\ &= (\mu - 1)^2 \vec{1}^T B(B^T B)^{-1} B^T \vec{1} \qquad B = D_x^T A \\ &\le (\mu - 1)^2 m \\ &= 1/4\end{aligned}$$

This means we are in the range of quadratic convergence!!!

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the number of Newton iterations per outer iteration is very small; in practise only 1 or 2

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We need $t_k = \mu^k t_0 \geq m/\epsilon$. This holds when

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We get a bound of

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We show how to get a starting point with $t_0 = 1/2^L$. Together with $\epsilon \approx 2^{-L}$ we get $\mathcal{O}(L\sqrt{m})$ iterations.

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Damped Newton Method

For $x \in P^\circ$ and direction $v \neq 0$ define

$$\sigma_x(v) := \max_i \frac{a_i^T v}{s_i(x)}$$

Observation:

$$x + \alpha v \in P \qquad \text{for } \alpha \in \{0, 1/\sigma_x(v)\}$$

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The following argument shows that f_t is well behaved. For small α the reduction of $f_t(x)$ is close to linear estimate.

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The following argument shows that f_t is well behaved. For small α the reduction of $f_t(x)$ is close to linear estimate.

$$f_t(x + \alpha v) - f_t(x) = tc^T \alpha v + \phi(x + \alpha v) - \phi(x)$$

$$\begin{aligned}\phi(x + \alpha v) - \phi(x) &= -\textstyle\sum_i \log(s_i(x + \alpha v)) + \sum_i \log(s_i(x)) \\ &= -\textstyle\sum_i \log(s_i(x + \alpha v)/s_i(x)) \\ &= -\textstyle\sum_i \log(1 - a_i^T \alpha v / s_i(x))\end{aligned}$$

Damped Newton Method

For $x \in P^\circ$ and direction $v \neq 0$ define

$$\sigma_x(v) := \max_i \frac{a_i^T v}{s_i(x)}$$

Observation:

$$x + \alpha v \in P \qquad \text{for } \alpha \in \{0, 1/\sigma_x(v)\}$$

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Define $w_i = a_i^T v / s_i(x)$ and $\sigma = \max_i w_i$. Then

$$f_t(x + \alpha v) - f_t(x) - \nabla f_t(x)^T \alpha v$$

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f_t(x+\alpha v) - f_t(x) - \nabla f_t(x)^T \alpha v
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Damped Newton Iteration:

In a damped Newton step we choose

$$x_+ = x + \frac{1}{1+\sigma_x(\Delta x_{\text{nt}})}\Delta x_{\text{nt}}$$

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Damped Newton Method

Theorem:
In a damped Newton step the cost decreases by at least

$$\lambda_t(x) - \log(1 + \lambda_t(x))$$

Proof: The decrease in cost is

$$-\alpha \nabla f_t(x)^T v + \frac{1}{\sigma^2} \|v\|_{H_x} (\alpha\sigma + \log(1 - \alpha\sigma))$$

Choosing $\alpha = \frac{1}{1+\sigma}$ and $v = \Delta x_{\text{nt}}$ gives

Damped Newton Method

$$\begin{aligned} &\le -\sum_i \frac{w_i^2}{\sigma^2}\Big(\alpha\sigma + \log(1 - \alpha\sigma)\Big) \\ &= -\frac{1}{\sigma^2}\|v\|_{H_x}^2\Big(\alpha\sigma + \log(1 - \alpha\sigma)\Big) \end{aligned}$$

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Damped Newton Method

$$\geq \lambda_t(x) - \log(1 + \lambda_t(x))$$
$$\geq 0.09$$

for $\lambda_t(x) \geq 0.5$

Centering Algorithm:

Input: precision δ; starting point x

1. compute Δx_{nt} and $\lambda_t(x)$
2. if $\lambda_t(x) \leq \delta$ return x
3. set $x := x + \alpha \Delta x_{\text{nt}}$ with

$$\alpha = \begin{cases} \frac{1}{1+\sigma_x(\Delta x_{\text{nt}})} & \lambda_t \geq 1/2 \\ 1 & \text{otw.} \end{cases}$$

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Centering

Lemma 56
The centering algorithm starting at x_0 reaches a point with $\lambda_t(x) \le \delta$ after

$$\frac{f_t(x_0) - \min_y f_t(y)}{0.09} + \mathcal{O}(\log\log(1/\delta))$$

iterations.

This can be very, very slow...

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How to get close to analytic center?

Let $P = \{Ax \le b\}$ be our (feasible) polyhedron, and x_0 a feasible point.

We change $b \to b + \frac{1}{\lambda} \cdot \vec{1}$, where $L = \langle A\rangle + \langle b\rangle + \langle c\rangle$ (encoding length) and $\lambda = 2^{2L}$. Recall that a basis is feasible in the old LP iff it is feasible in the new LP.

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Lemma [without proof]
The inverse of a matrix M can be represented with rational numbers that have denominators $z_{ij} = \det(M)$.

For two basis solutions $x_B, x_{\bar{B}}$, the cost-difference $c^T x_B - c^T x_{\bar{B}}$ can be represented by a rational number that has denominator $z = \det(A_B) \cdot \det(A_{\bar{B}}) \cdot \lambda$.

This means that in the perturbed LP it is sufficient to decrease the duality gap to $1/2^{4L}$ (i.e., $t \approx 2^{4L}$). This means the previous analysis essentially also works for the perturbed LP.

For a point x from the polytope (not necessarily BFS) the objective value $\bar{c}^T x$ is at most $n2^M 2^L$, where $M \le L$ is the encoding length of the largest entry in $\bar{c}$.

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How to get close to analytic center?

Start at x_0.

Choose $\hat{c} := -\nabla\phi(x)$.

$x_0 = x^*(1)$ is point on central path for $\hat{c}$ and $t = 1$.

You can travel the central path in both directions. Go towards 0 until $t \approx 1/2^{\Omega(L)}$. This requires $O(\sqrt{m}L)$ outer iterations.

Let $x_{\hat{c}}$ denote this point.

Let x_c denote the point that minimizes

$$t \cdot c^T x + \phi(x)$$

(i.e., same value for t but different c, hence, different central path).

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Part III

Approximation Algorithms

There are many practically important optimization problems that are NP-hard.

What can we do?

- Heuristics.
- Exploit special structure of instances occurring in practise.
- Consider algorithms that do not compute the optimal solution but provide solutions that are close to optimum.

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An α-approximation for an optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor of α of the value of an optimal solution.

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- We need algorithms for hard problems.
- It gives a rigorous mathematical base for studying heuristics.
- It provides a metric to compare the difficulty of various optimization problems.
- Proving theorems may give a deeper theoretical understanding which in turn leads to new algorithmic approaches.

Why not?

- Sometimes the results are very pessimistic due to the fact that an algorithm has to provide a close-to-optimum solution on every instance.

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An optimization problem $P = (\mathcal{I}, \mathrm{sol}, m, \mathrm{goal})$ is in **NPO** if

- $x \in \mathcal{I}$ can be decided in polynomial time
- $y \in \mathrm{sol}(\mathcal{I})$ can be verified in polynomial time
- m can be computed in polynomial time
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In other words: the decision problem is there a solution y with $m(x, y)$ at most/at least z is in NP.

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- y is candidate solution
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Definition 59 (Performance Ratio)

$$R(x,y) := \max\left\{\frac{m(x,y)}{m^*(x)}, \frac{m^*(x)}{m(x,y)}\right\}$$

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An algorithm A is an r-approximation algorithm iff

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A PTAS for a problem P from NPO is an algorithm that takes as input $x \in \mathcal{I}$ and $\epsilon > 0$ and produces a solution y for x with

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Definition 63 (APX – approximable)

A problem P from NPO is in APX if there exist a constant $r \geq 1$ and an r-approximation algorithm for P.

constant factor approximation...

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Problems that are in APX

MAXCUT. Given a graph $G = (V, E)$; partition V into two disjoint pieces A and B s.t. the number of edges between both pieces is maximized.

MAX-3SAT. Given a 3CNF-formula. Find an assignment to the variables that satisfies the maximum number of clauses.

Definition 63 (APX – approximable)

A problem P from NPO is in APX if there exist a constant $r \geq 1$ and an r-approximation algorithm for P.

constant factor approximation...

Problems with polylogarithmic approximation guarantees

- Set Cover
- Minimum Multicut
- Sparsest Cut
- Minimum Bisection

There is an r-approximation with $r \leq \mathcal{O}(\log^c(|x|))$ for some constant c.

Note that only for some of the above problem a matching lower bound is known.

Problems that are in APX

MAXCUT. Given a graph $G = (V, E)$; partition V into two disjoint pieces A and B s.t. the number of edges between both pieces is maximized.

MAX-3SAT. Given a 3CNF-formula. Find an assignment to the variables that satisfies the maximum number of clauses.

There are really difficult problems!

Theorem 64
For any constant $\epsilon > 0$ there does not exist an $\Omega(n^{1-\epsilon})$-approximation algorithm for the maximum clique problem on a given graph G with n nodes unless $P = NP$.

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There are weird problems!

Asymmetric k-Center admits an $\mathcal{O}(\log^* n)$-approximation.

There is no $o(\log^* n)$-approximation to Asymmetric k-Center unless $NP \subseteq DTIME(n^{\log\log\log n})$.

There are really difficult problems!

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Note that an n-approximation is trivial.

Class APX not important in practise.

Instead of saying problem P is in APX one says problem P admits a 4-approximation.

One only says that a problem is APX-hard.

There are weird problems!
Asymmetric k-Center admits an $\mathcal{O}(\log^* n)$-approximation.

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A Mixed Integer Program is a Linear Program in which a subset of the variables are required to be integral.

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Given a ground set U, a collection of subsets $S_1, \dots, S_k \subseteq U$, where the i-th subset S_i has weight/cost w_i. Find a collection $I \subseteq \{1, \dots, k\}$ such that

$$\forall u \in U \exists i \in I : \ u \in S_i \ \text{(every element is covered)}$$

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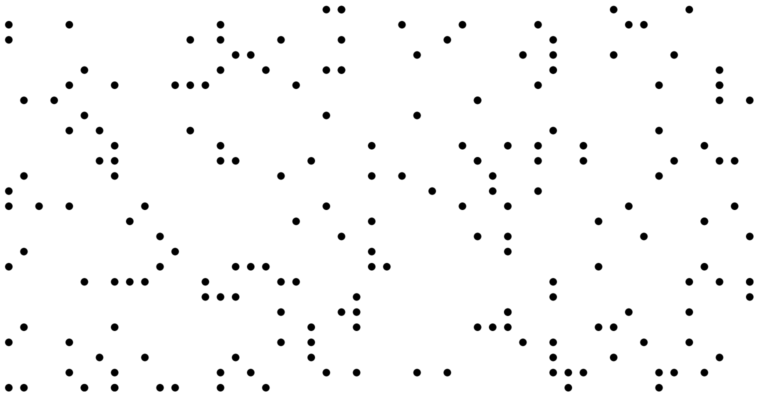

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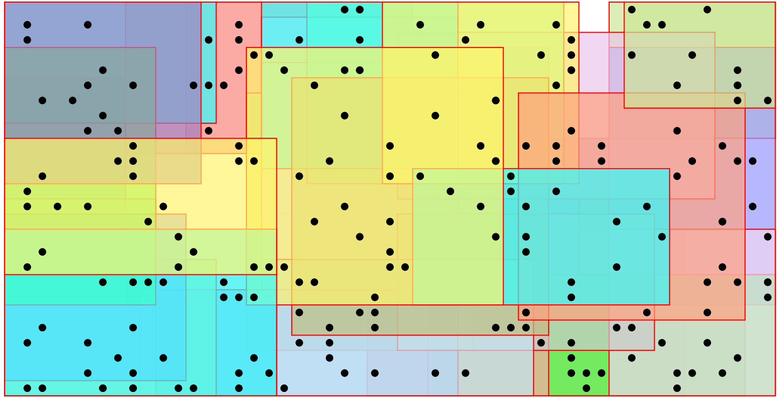

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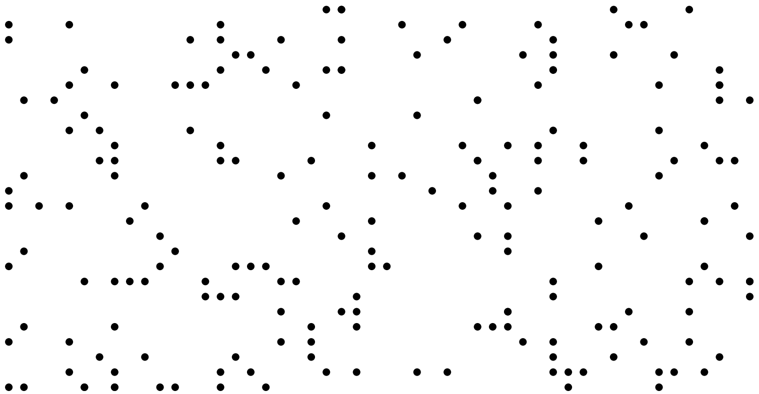

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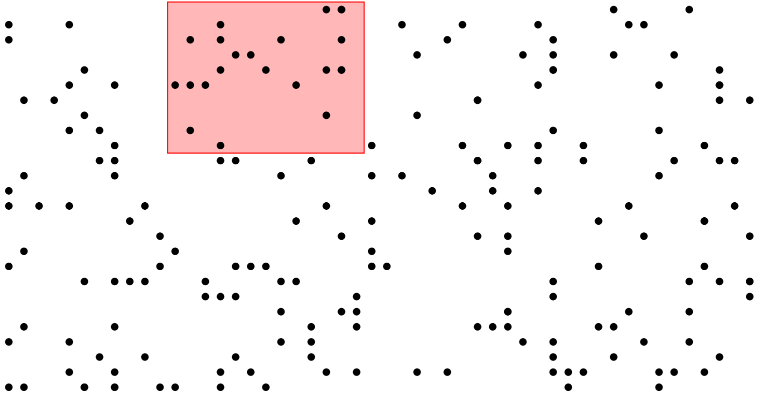

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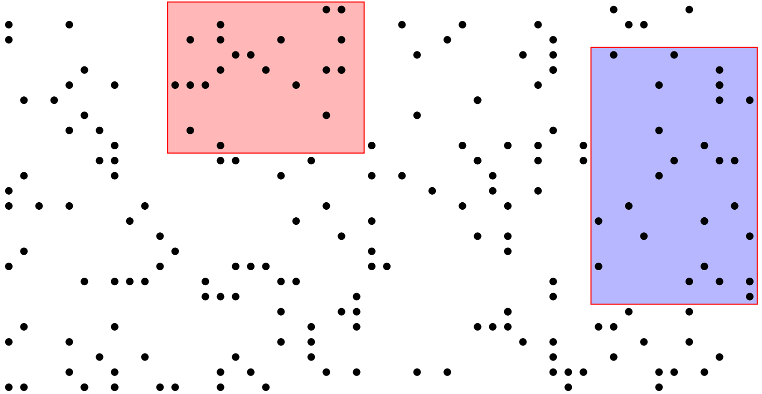

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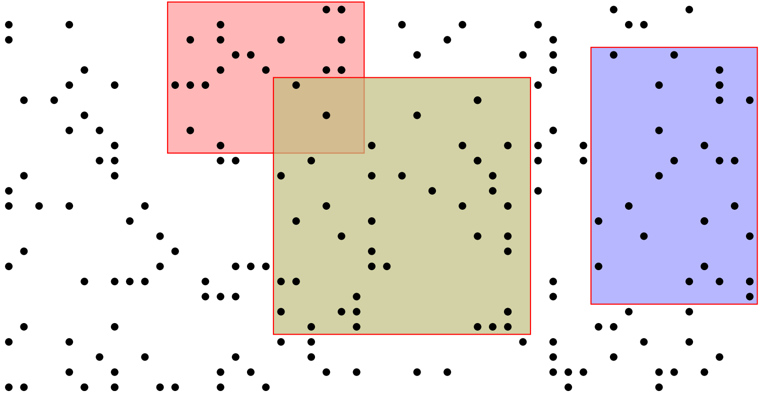

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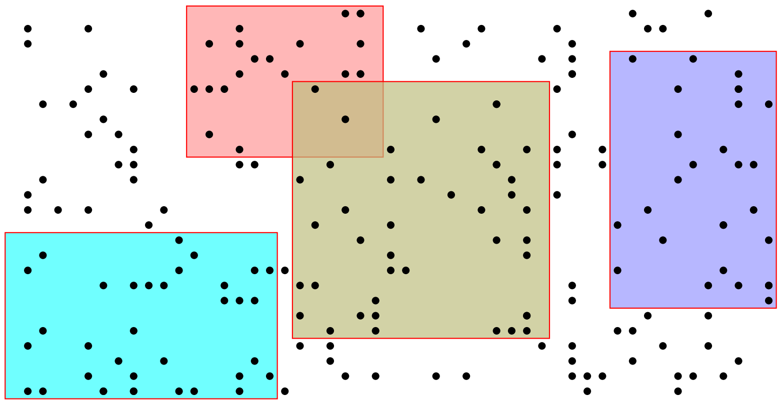

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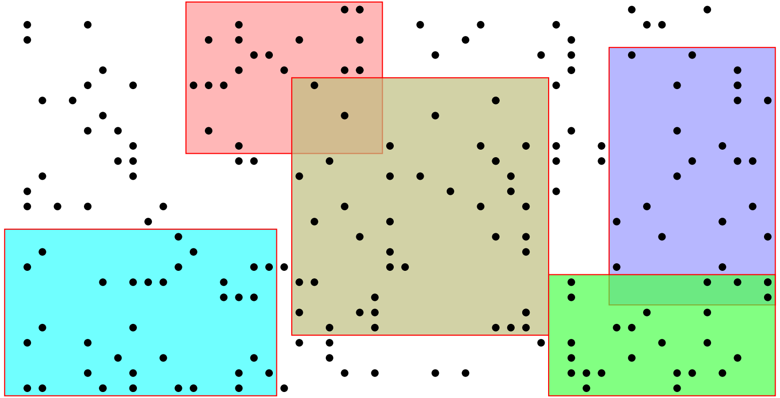

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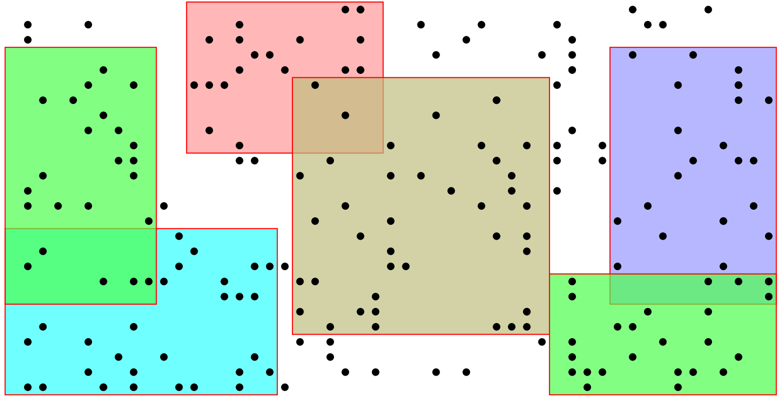

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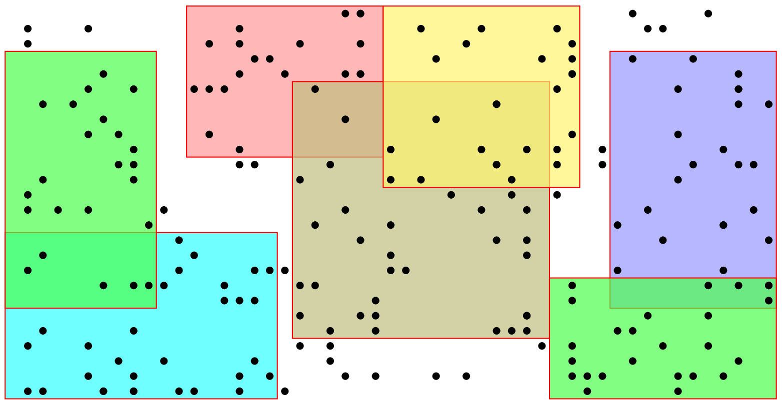

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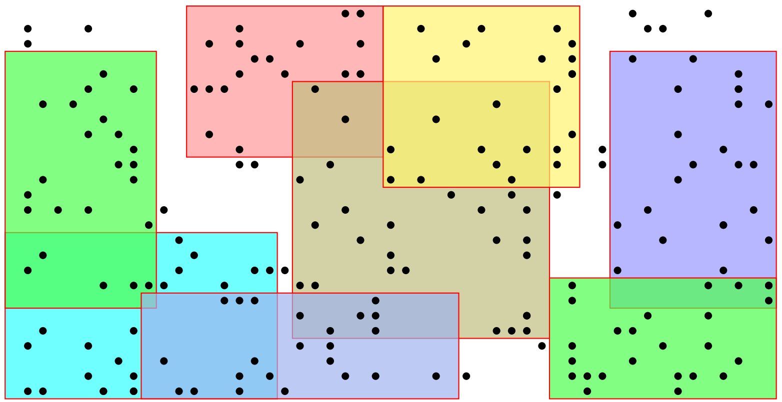

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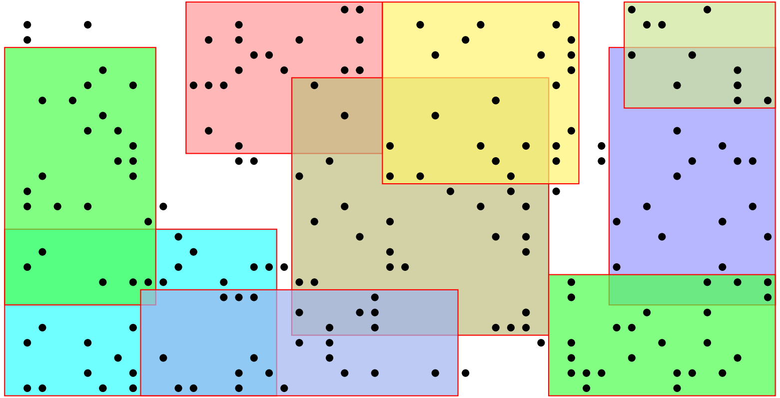

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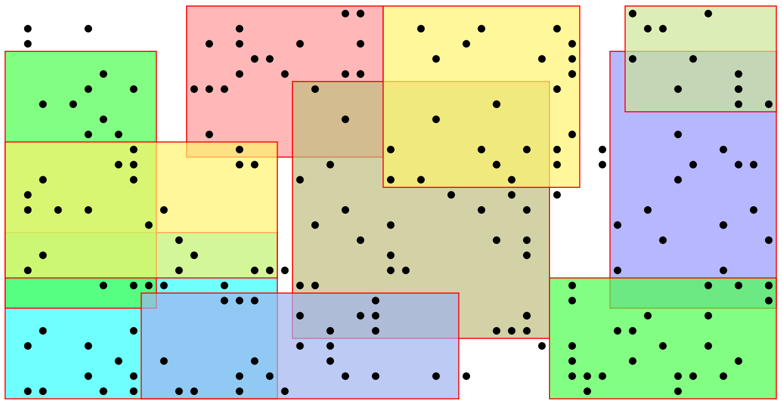

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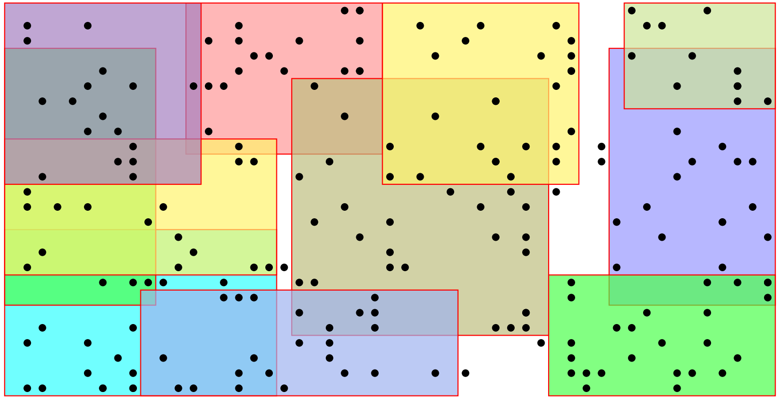

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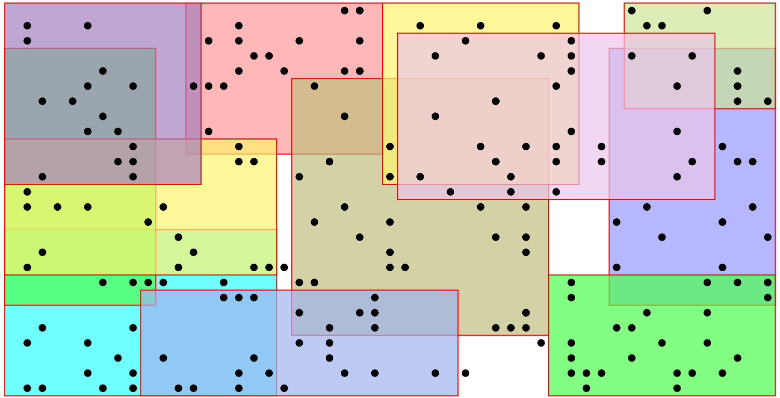

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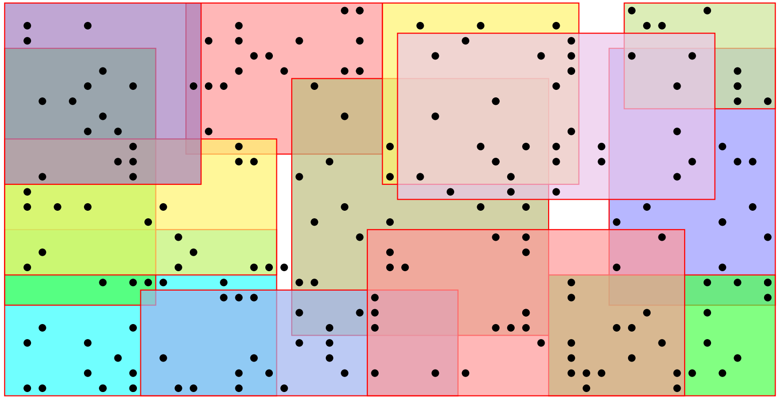

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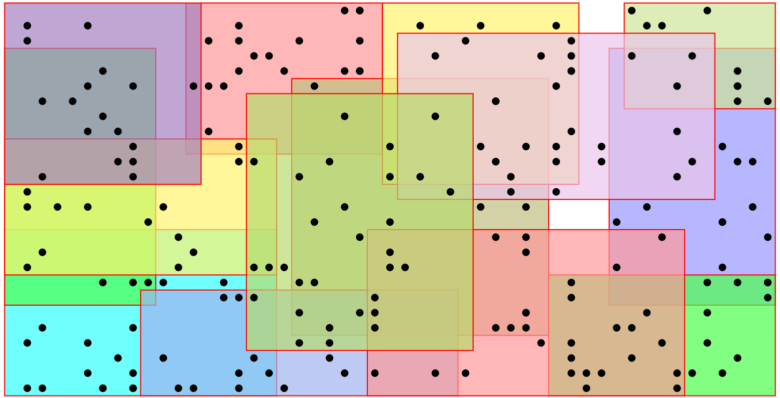

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IP-Formulation of Set Cover

$$\begin{array}{llrl} \min & & \sum_i w_i x_i & \\ \text{s.t.} & \forall u \in U & \sum_{i:u\in S_i} x_i & \geq 1 \\ & \forall i \in \{1,\dots,k\} & x_i & \geq 0 \\ & \forall i \in \{1,\dots,k\} & x_i & \text{integral} \end{array}$$

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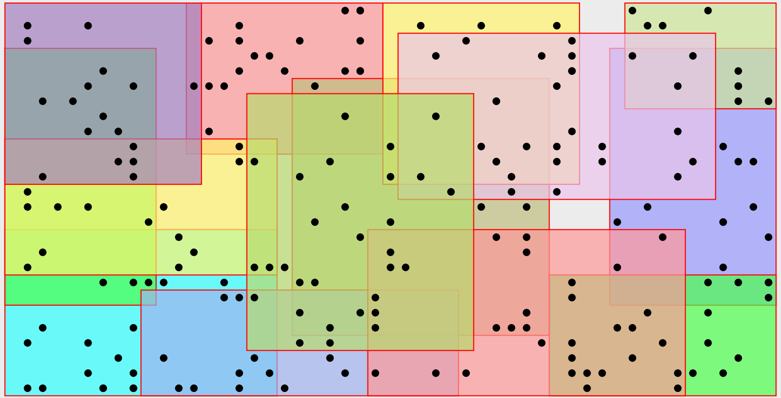

Vertex Cover

Given a graph $G = (V, E)$ and a weight w_v for every node. Find a vertex subset $S \subseteq V$ of minimum weight such that every edge is incident to at least one vertex in S.

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\min & & \sum_{v\in V} w_v x_v & & \\
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Given a graph $G=(V,E)$ and a weight w_v for every node. Find a vertex subset $S \subseteq V$ of minimum weight such that every edge is incident to at least one vertex in S.

Maximum Weighted Matching

Given a graph $G = (V, E)$, and a weight w_e for every edge $e \in E$. Find a subset of edges of maximum weight such that no vertex is incident to more than one edge.

$$\begin{array}{llrcl} \max & & \sum_{e\in E} w_e x_e & & \\ \text{s.t.} & \forall v \in V & \sum_{e:v\in e} x_e & \leq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

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Maximum Independent Set

Given a graph $G = (V, E)$, and a weight w_v for every node $v \in V$. Find a subset $S \subseteq V$ of nodes of maximum weight such that no two vertices in S are adjacent.

$$\begin{array}{llrcl} \max & & \sum_{v\in V} w_v x_v & & \\ \text{s.t.} & \forall e=(i,j)\in E & x_i + x_j & \le & 1 \\ & \forall v \in V & x_v & \in & \{0,1\} \end{array}$$

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Knapsack

Given a set of items $\{1,\dots,n\}$, where the i-th item has weight w_i and profit p_i, and given a threshold K. Find a subset $I \subseteq \{1,\dots,n\}$ of items of total weight at most K such that the profit is maximized.

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Relaxations

Definition 67

A linear program LP is a relaxation of an integer program IP if any feasible solution for IP is also feasible for LP and if the objective values of these solutions are identical in both programs.

We obtain a relaxation for all examples by writing $x_i \in [0,1]$ instead of $x_i \in \{0,1\}$.

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Relaxations

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A linear program LP is a relaxation of an integer program IP if any feasible solution for IP is also feasible for LP and if the objective values of these solutions are identical in both programs.

We obtain a relaxation for all examples by writing $x_i \in [0,1]$ instead of $x_i \in \{0,1\}$.

Knapsack

Given a set of items $\{1,\ldots,n\}$, where the i-th item has weight w_i and profit p_i, and given a threshold K. Find a subset $I \subseteq \{1,\ldots,n\}$ of items of total weight at most K such that the profit is maximized.

$$\begin{array}{lrcl} \max & \sum_{i=1}^{n} p_i x_i & & \\ \text{s.t.} & \sum_{i=1}^{n} w_i x_i & \leq & K \\ \forall i \in \{1,\ldots,n\} & x_i & \in & \{0,1\} \end{array}$$

By solving a relaxation we obtain an upper bound for a maximization problem and a lower bound for a minimization problem.

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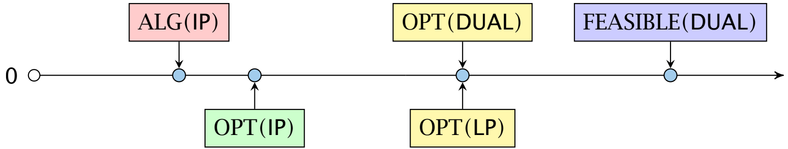
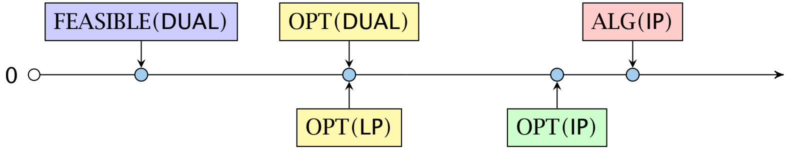
Relations

Maximization Problems:



Minimization Problems:



By solving a relaxation we obtain an upper bound for a maximization problem and a lower bound for a minimization problem.

EADS II
Harald Räcke
12 Integer Programs

Technique 1: Round the LP solution.

We first solve the LP-relaxation and then we round the fractional values so that we obtain an integral solution.

Set Cover relaxation:

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\begin{array}{llrcl}
\min & & \sum_{i=1}^{k} w_i x_i & & \\
\text{s.t.} & \forall u \in U & \sum_{i:u \in S_i} x_i & \geq & 1 \\
 & \forall i \in \{1,\dots,k\} & x_i & \in & [0,1]
\end{array}
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Let f_u be the number of sets that the element u is contained in (the frequency of u). Let $f = \max_u \{f_u\}$ be the maximum frequency.

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Rounding Algorithm:
Set all x_i-values with $x_i \geq \frac{1}{f}$ to 1. Set all other x_i-values to 0.

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Lemma 68

The rounding algorithm gives an f-approximation.

Proof: Every $u \in U$ is covered.

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- We know that $\sum_{i:u \in S_i} x_i \geq 1$.
- The sum contains at most $f_u \leq f$ elements.
- Therefore one of the sets that contain u must have $x_i \geq 1/f$.
- This set will be selected. Hence, u is covered.

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Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\begin{array}{lrl} \min & \sum_{i\in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i:u\in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

Dual:

$$\begin{array}{lrl} \max & \sum_{u\in U} y_u & \\ \text{s.t. } \forall i & \sum_{u:u\in S_i} y_u & \leq w_i \\ & y_u & \geq 0 \end{array}$$

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$$\begin{aligned} \sum_{i\in I} w_i &\leq \sum_{i=1}^{k} w_i (f \cdot x_i) \\ &= f \cdot \mathrm{cost}(x) \\ &\leq f \cdot \mathrm{OPT}\ . \end{aligned}$$

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Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

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Lemma 69

The resulting index set is an f-approximation.

Proof:

Every $u \in U$ is covered.

- Suppose there is a u that is not covered.
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$$= \sum_u |\{i \in I : u \in S_i\}| \cdot y_u$$

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Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means I' is never better than I.

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Technique 3: The Primal Dual Method

The previous two rounding algorithms have the disadvantage that it is necessary to solve the LP. The following method also gives an f-approximation without solving the LP.

For estimating the cost of the solution we only required two properties.

- The solution is dual feasible and, hence,

$$\sum_u y_u \le \text{cost}(x^*) \le \text{OPT}$$

  where x^* is an optimum solution to the primal LP.

- The set I contains only sets for which the dual inequality is tight.

Of course, we also need that I is a cover.

Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means I' is never better than I.

- Suppose that we take S_i in the first algorithm. I.e., $i \in I$.
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Algorithm 1 PrimalDual

1: $y \leftarrow 0$
2: $I \leftarrow \emptyset$
3: **while** exists $u \notin \bigcup_{i \in I} S_i$ **do**
4: increase dual variable y_u until constraint for some new set S_ℓ becomes tight
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$\hat{S}_j \leftarrow S_j$ for all j
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In every round the Greedy algorithm takes the set that covers remaining elements in the most cost-effective way.

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Lemma 70
Given positive numbers $a_1, \dots, a_k$ *and* $b_1, \dots, b_k$, *and* $S \subseteq \{1, \dots, k\}$ *then*

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Let n_ℓ denote the number of elements that remain at the beginning of iteration ℓ. $n_1 = n = |U|$ and $n_{s+1} = 0$ if we need s iterations.

In the ℓ-th iteration

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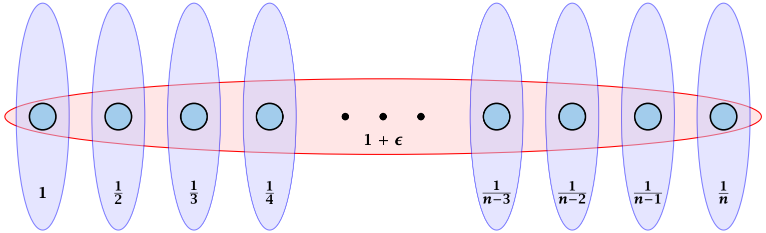
A tight example:



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$$
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Technique 5: Randomized Rounding

One round of randomized rounding:
Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

Version A: Repeat rounds until you nearly have a cover. Cover remaining elements by some simple heuristic.

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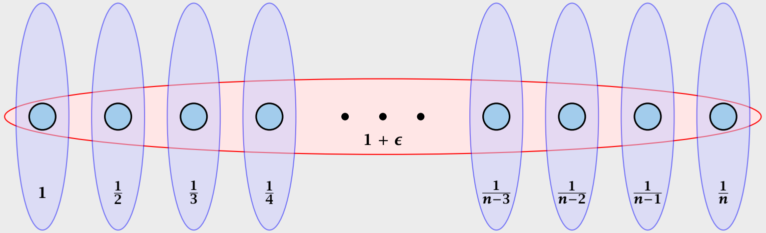

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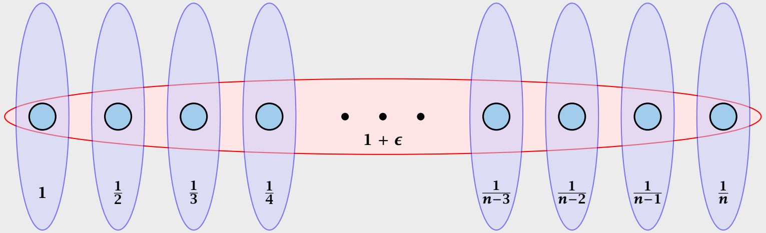

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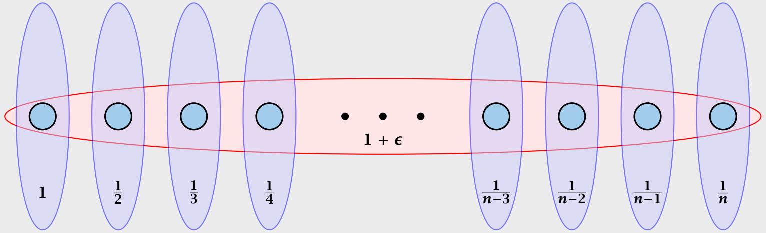

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Probability that $u \in U$ is not covered (after ℓ rounds):

$$\Pr[u \text{ not covered after } \ell \text{ round}] \le \frac{1}{e^\ell} \enspace .$$

Technique 5: Randomized Rounding

One round of randomized rounding:
Pick set S_j uniformly at random with probability $1-x_j$ (for all j).

Version A: Repeat rounds until you nearly have a cover. Cover remaining elements by some simple heuristic.

Version B: Repeat for s rounds. If you have a cover STOP. Otherwise, repeat the whole algorithm.

Probability that $u \in U$ is not covered (in one round):

$$\begin{aligned}\Pr[u \text{ not covered in one round}] \\ &= \prod_{j:u\in S_j}(1-x_j) \le \prod_{j:u\in S_j} e^{-x_j} \\ &= e^{-\sum_{j:u\in S_j} x_j} \le e^{-1} \ .\end{aligned}$$

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$\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}]$

Probability that $u \in U$ is not covered (in one round):

$$
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$$= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \ldots \vee u_n \text{ not covered}]$$

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Lemma 71

With high probability $\mathcal{O}(\log n)$ *rounds suffice.*

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Lemma 71
With high probability $\mathcal{O}(\log n)$ rounds suffice.

With high probability:
For any constant α the number of rounds is at most $\mathcal{O}(\log n)$ with probability at least $1 - n^{-\alpha}$.

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Probability that $u \in U$ is not covered (after ℓ rounds):

$$\Pr[u \text{ not covered after } \ell \text{ round}] \le \frac{1}{e^{\ell}}\ .$$

Proof: We have

$$\Pr[\#\text{rounds} \geq (\alpha+1)\ln n] \leq ne^{-(\alpha+1)\ln n} = n^{-\alpha} .$$

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Randomized rounding gives an $\mathcal{O}(\log n)$ approximation. The running time is polynomial with high probability.

Theorem 72 (without proof)

There is no approximation algorithm for set cover with approximation guarantee better than $\frac{1}{2}\log n$ unless NP has quasi-polynomial time algorithms (algorithms with running time $2^{\text{poly}(\log n)}$).

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Integrality Gap

The integrality gap of the SetCover LP is $\Omega(\log n)$.

- $n = 2^k - 1$
- Elements are all vectors $\vec{x}$ over $GF[2]$ of length k (excluding zero vector).
- Every vector $\vec{y}$ defines a set as follows

$$S_{\vec{y}} := \{\vec{x} \mid \vec{x}^T\vec{y} = 1\}$$

- each set contains 2^{k-1} vectors; each vector is contained in 2^{k-1} sets
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Every collection of $p < k$ sets does not cover all elements.

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Techniques:

- Deterministic Rounding
- Rounding of the Dual
- Primal Dual
- Greedy
- Randomized Rounding
- Local Search
- Rounding Data + Dynamic Programming

Integrality Gap

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Scheduling Jobs on Identical Parallel Machines

Given n jobs, where job $j \in \{1, \dots, n\}$ has processing time p_j. Schedule the jobs on m identical parallel machines such that the Makespan (finishing time of the last job) is minimized.

$$
\begin{array}{llrcl}
\min & & & & L \\
\text{s.t.} & \forall \text{machines } i & \sum_j p_j \cdot x_{j,i} & \leq & L \\
& \forall \text{jobs } j & \sum_i x_{j,i} \geq 1 & & \\
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Here the variable $x_{j,i}$ is the decision variable that describes whether job j is assigned to machine i.

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Lower Bounds on the Solution

Let for a given schedule C_j denote the finishing time of machine j, and let $C_{\max}$ be the makespan.

Let $C^*_{\max}$ denote the makespan of an optimal solution.

Clearly

$$C^*_{\max} \geq \max_j p_j$$

as the longest job needs to be scheduled somewhere.

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A local search algorithm successively makes certain small (cost/profit improving) changes to a solution until it does not find such changes anymore.

It is conceptionally very different from a Greedy algorithm as a feasible solution is always maintained.

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Local Search for Scheduling

Local Search Strategy: Take the job that finishes last and try to move it to another machine. If there is such a move that reduces the makespan, perform the switch.

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A local search algorithm successively makes certain small (cost/profit improving) changes to a solution until it does not find such changes anymore.

It is conceptionally very different from a Greedy algorithm as a feasible solution is always maintained.

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The interval $[S_\ell, C_\ell]$ is of length $p_\ell \leq C^*_{\max}$.

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$$p_\ell \approx S_\ell + \frac{S_\ell}{m-1}$$

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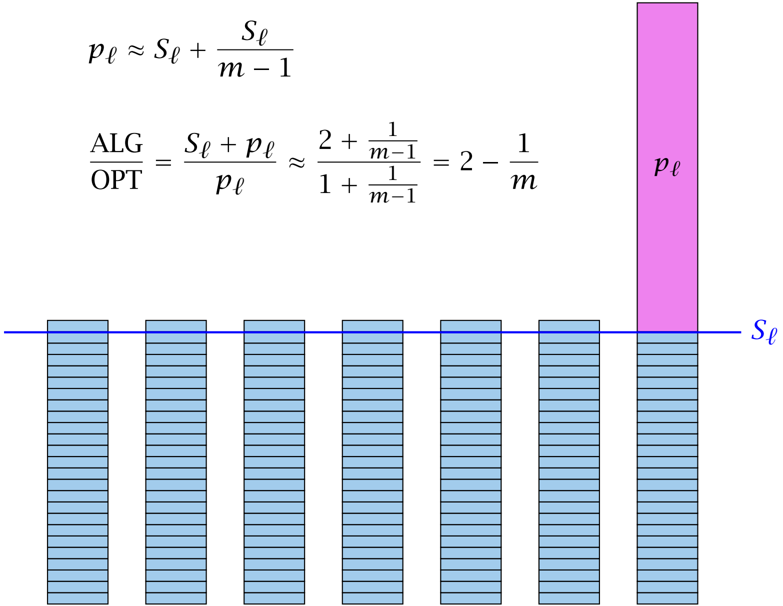

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List Scheduling:
Order all processes in a list. When a machine runs empty assign the next yet unprocessed job to it.

Alternatively:
Consider processes in some order. Assign the i-th process to the least loaded machine.

It is easy to see that the result of these greedy strategies fulfill the local optimally condition of our local search algorithm. Hence, these also give 2-approximations.

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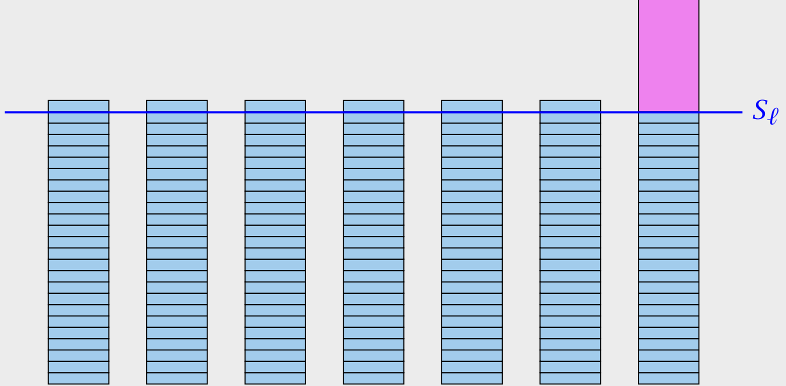

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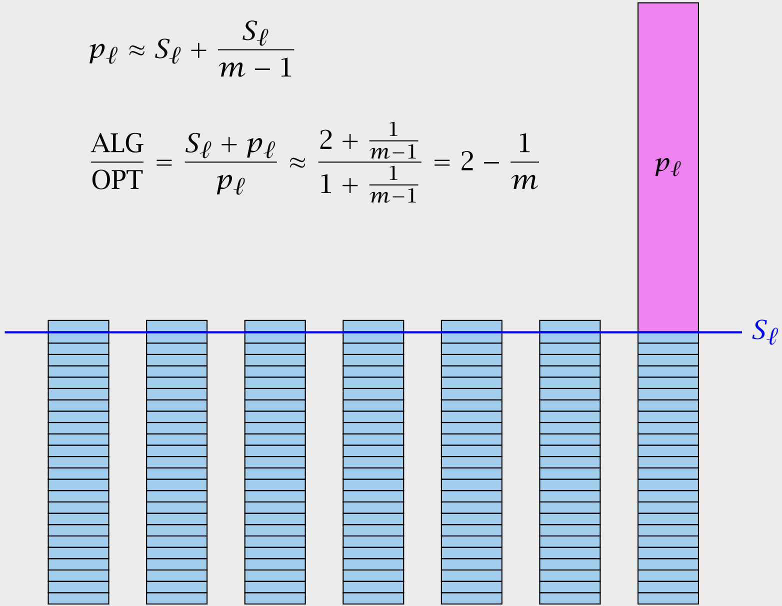

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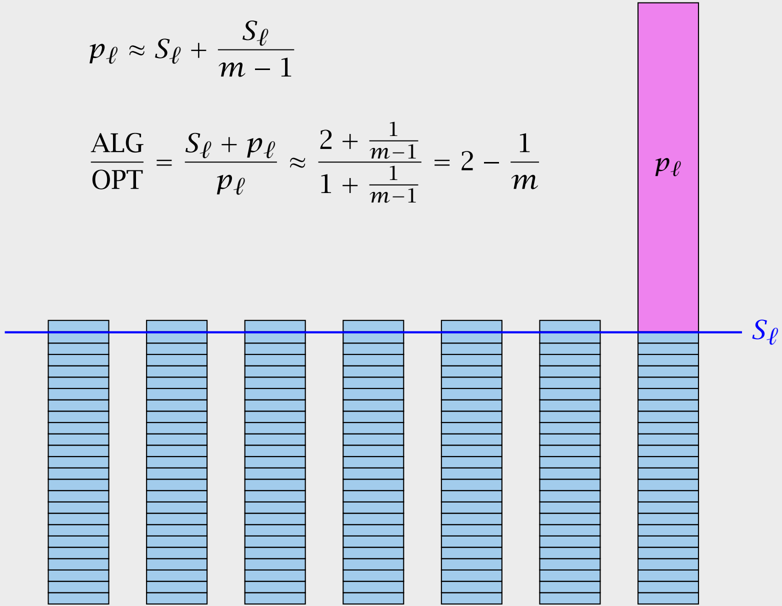

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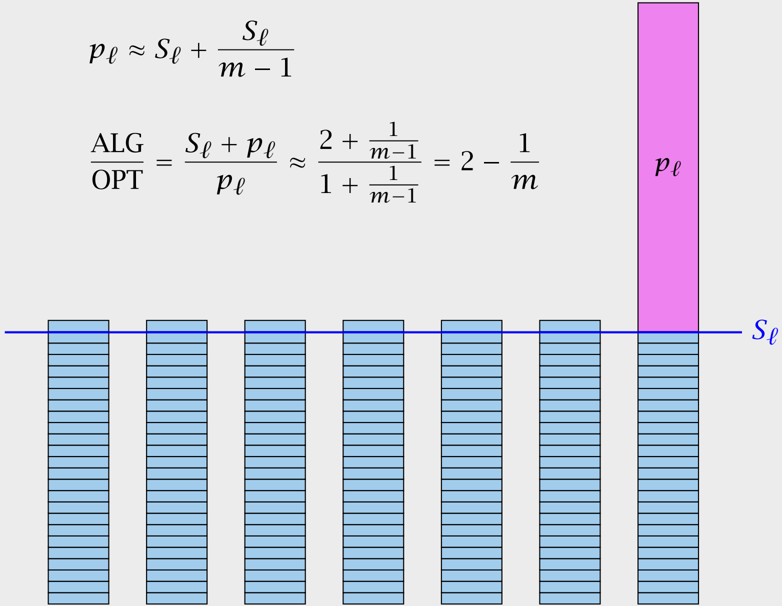

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If we order the list according to non-increasing processing times the approximation guarantee of the list scheduling strategy improves to $4/3$.

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Proof:

- Let $p_1 \geq \cdots \geq p_n$ denote the processing times of a set of jobs that form a counter-example.
- Wlog. the last job to finish is n (otw. deleting this job gives another counter-example with fewer jobs).
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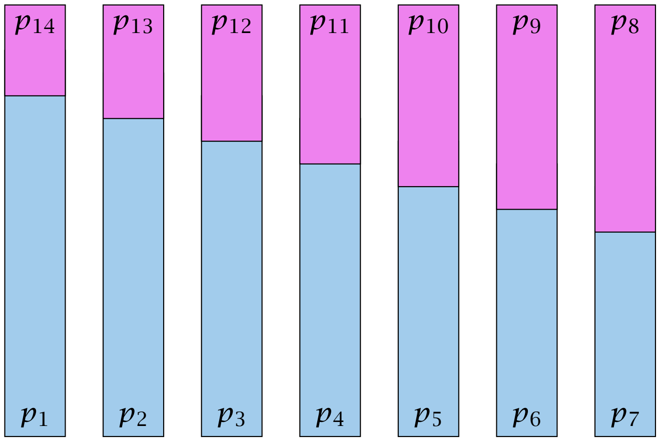
When in an optimal solution a machine can have at most 2 jobs the optimal solution looks as follows.



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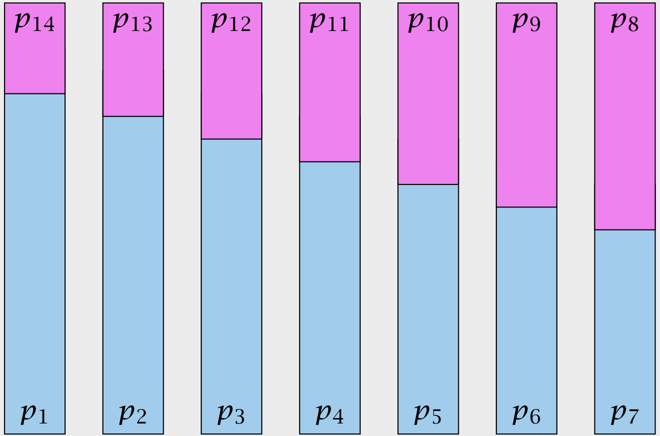

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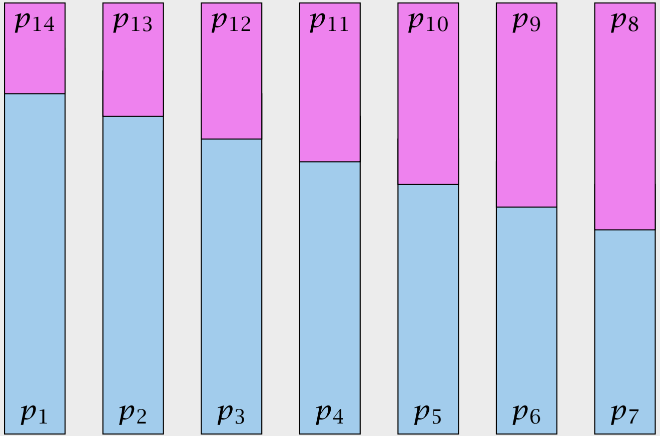

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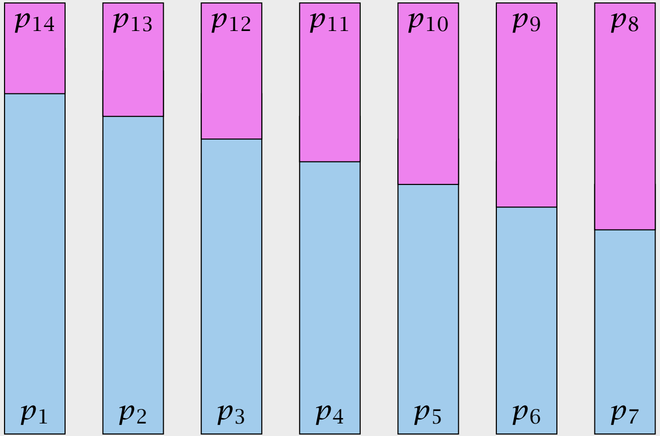

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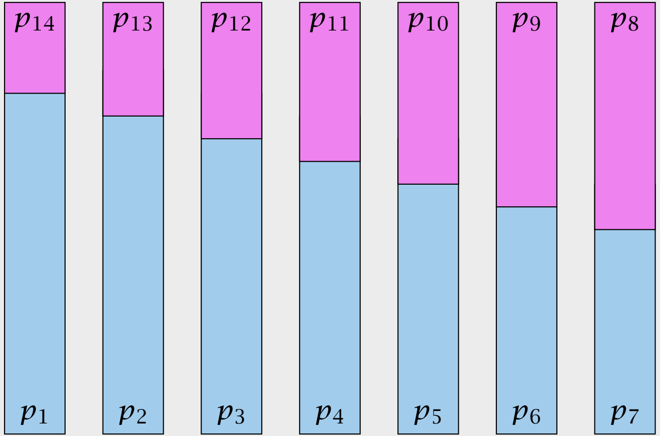

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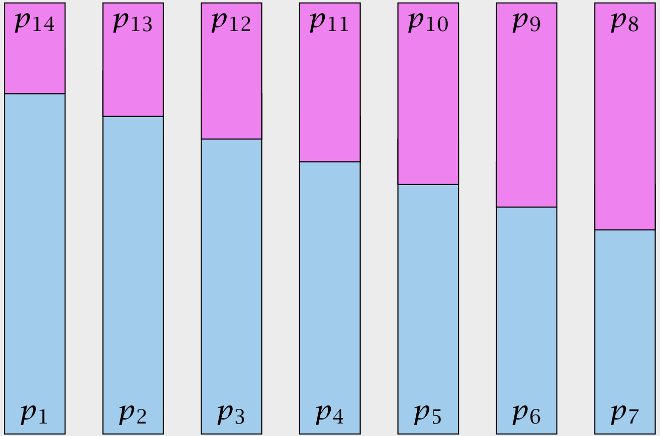

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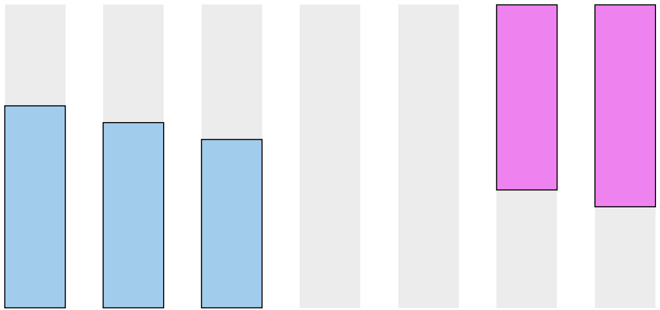

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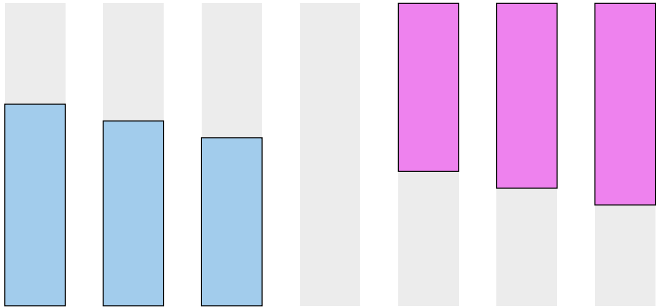

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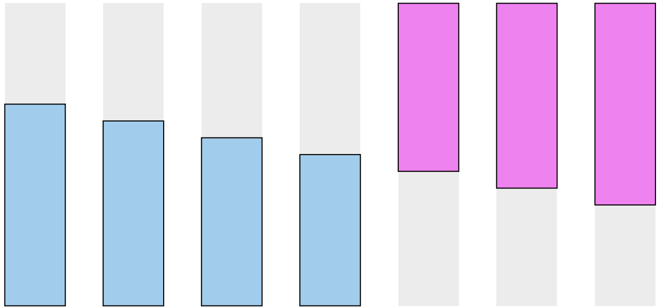

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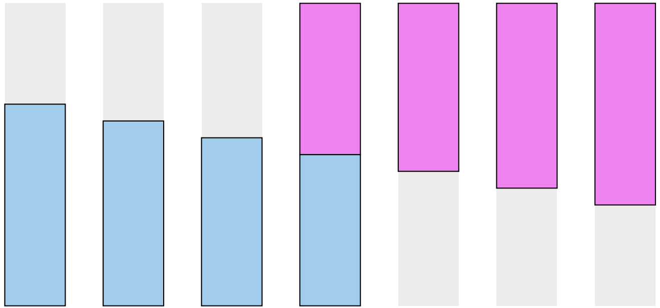

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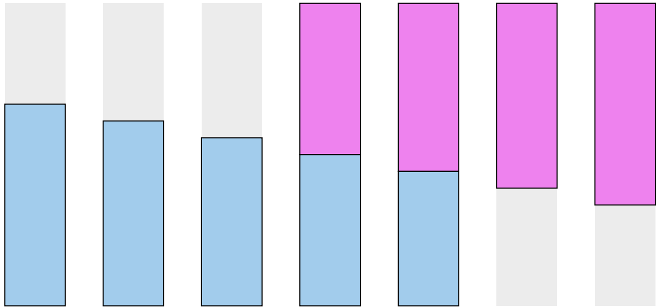

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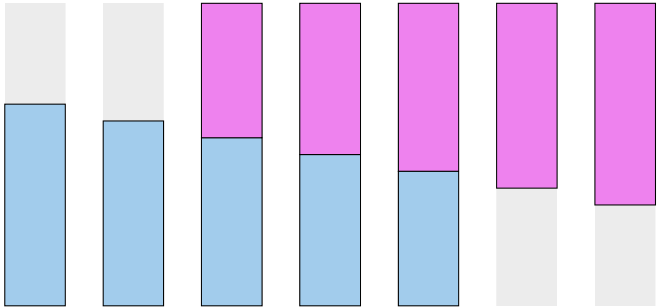

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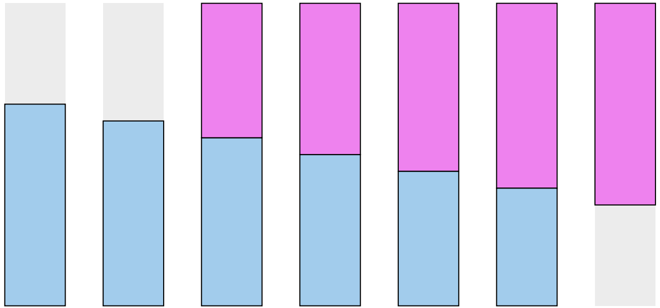

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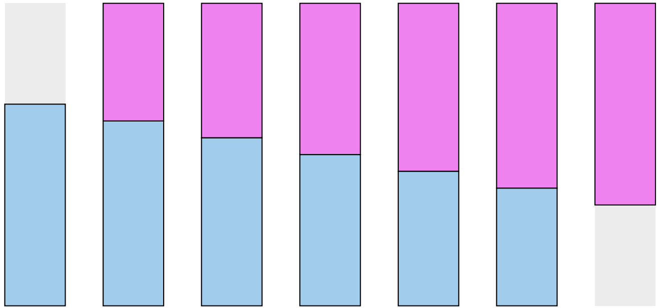

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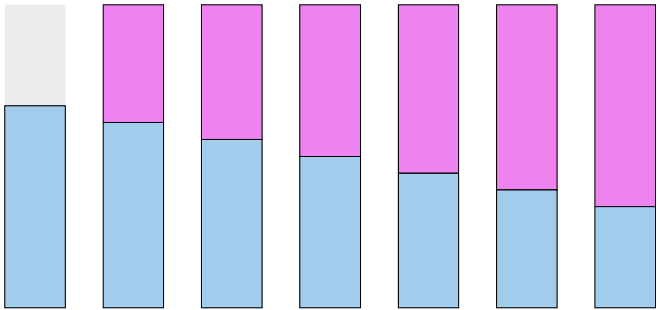

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EADS II
Harald Räcke
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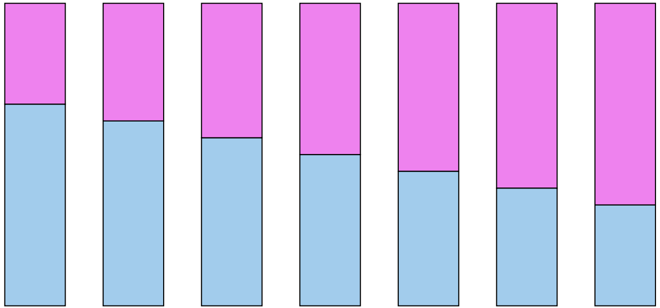

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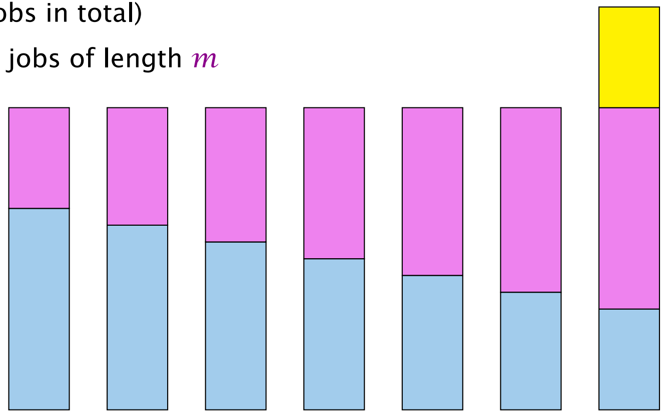

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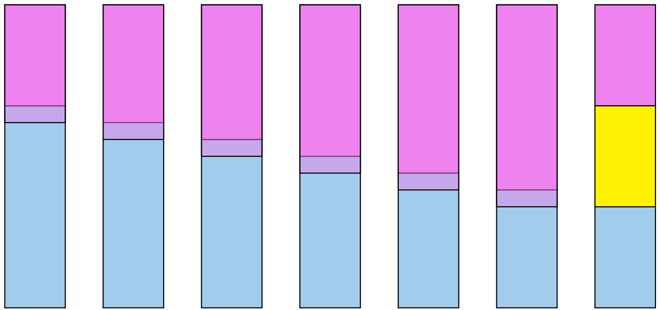

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Lemma 75
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Proof:

- Take the optimum TSP-tour.
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- Start with a tour on a subset S containing a single node.
- Take the node v closest to S. Add it S and expand the existing tour on S to include v.
- Repeat until all nodes have been processed.

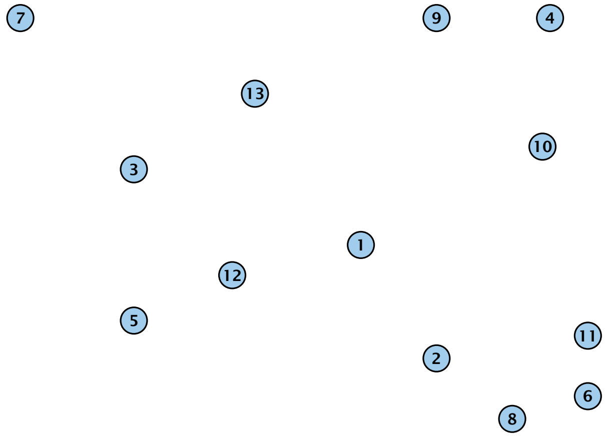
TSP: Lower Bound I

Lemma 75
The cost $\mathrm{OPT}_{TSP}(G)$ *of an optimum traveling salesman tour is at least as large as the weight* $\mathrm{OPT}_{MST}(G)$ *of a minimum spanning tree in* G.

Proof:

- Take the optimum TSP-tour.
- Delete one edge.
- This gives a spanning tree of cost at most $\mathrm{OPT}_{\mathrm{TSP}}(G)$.

TSP: Greedy Algorithm



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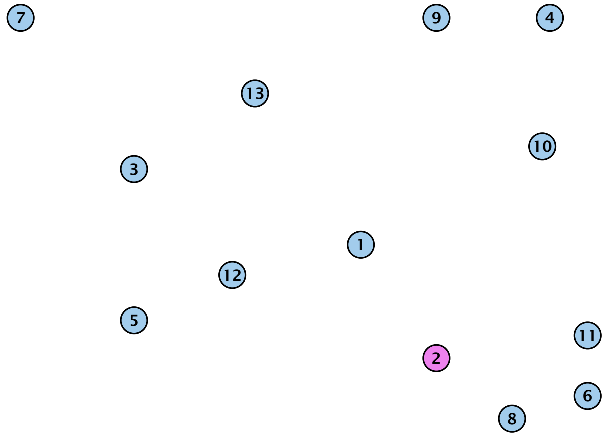

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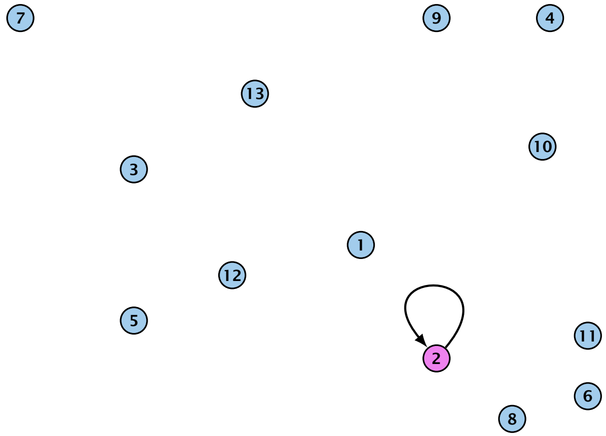

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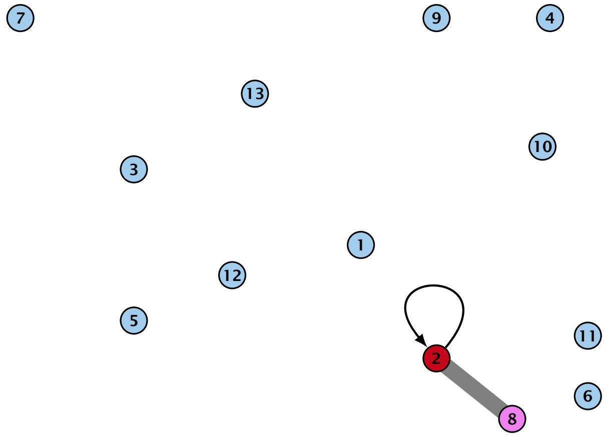

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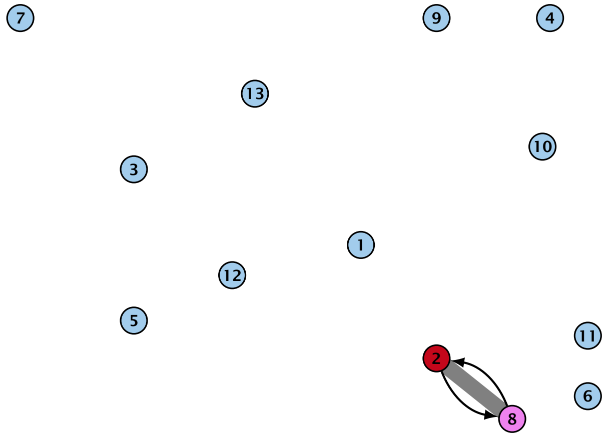

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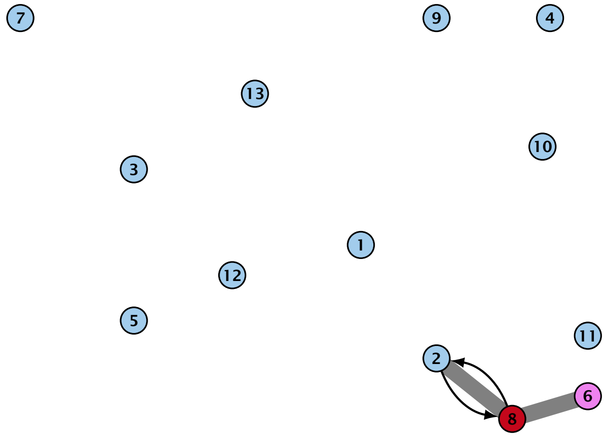

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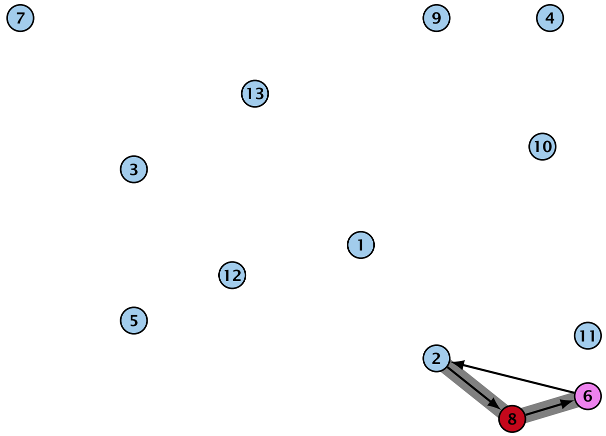

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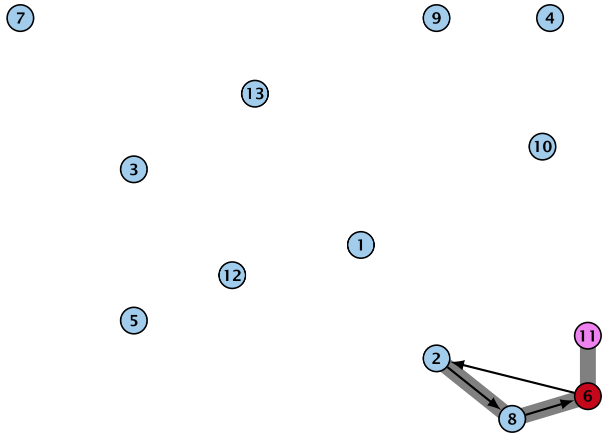

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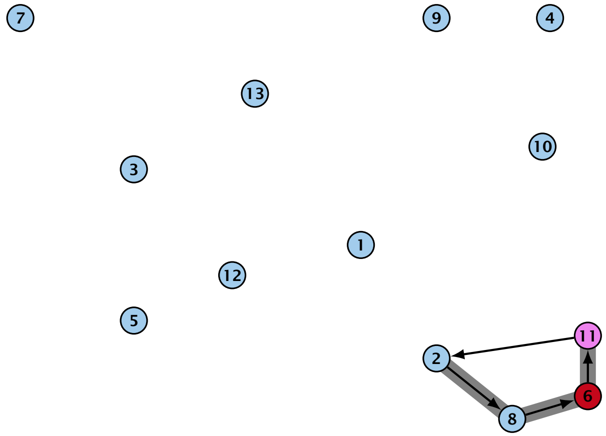

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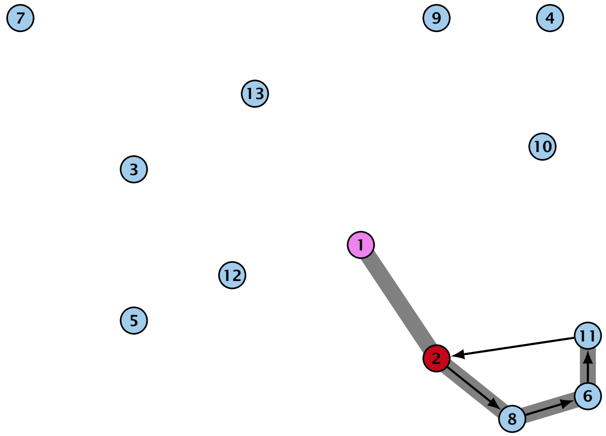

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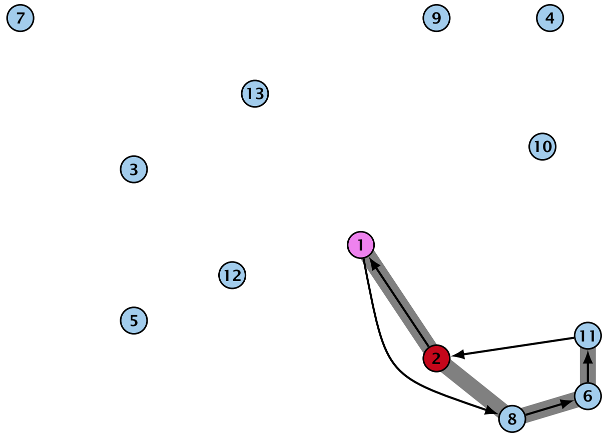

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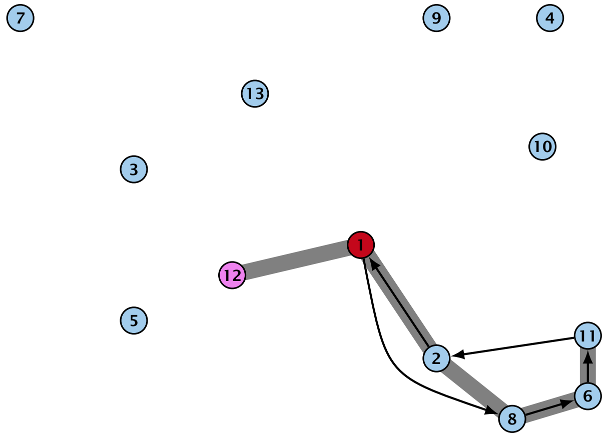

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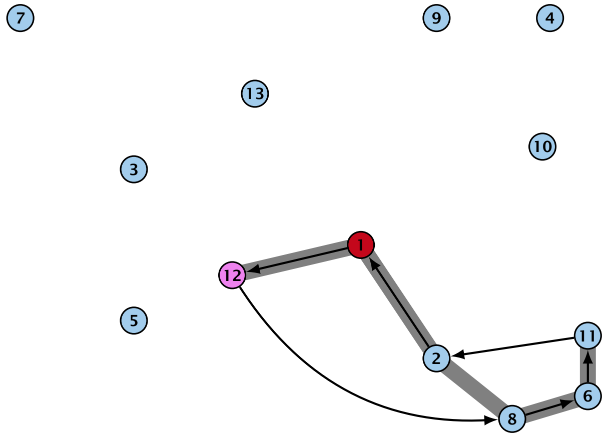

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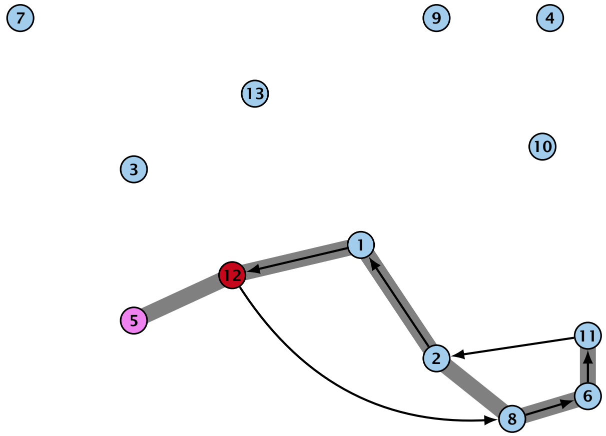

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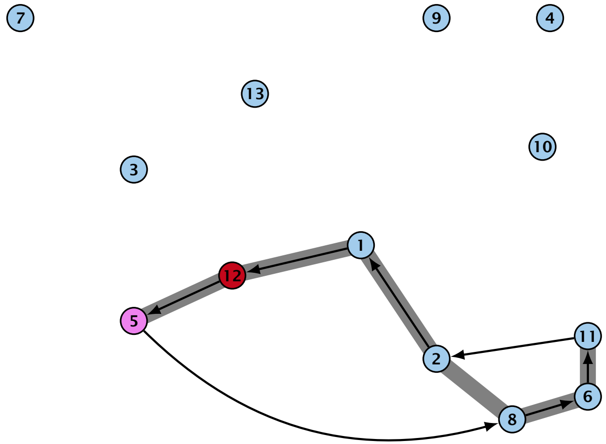

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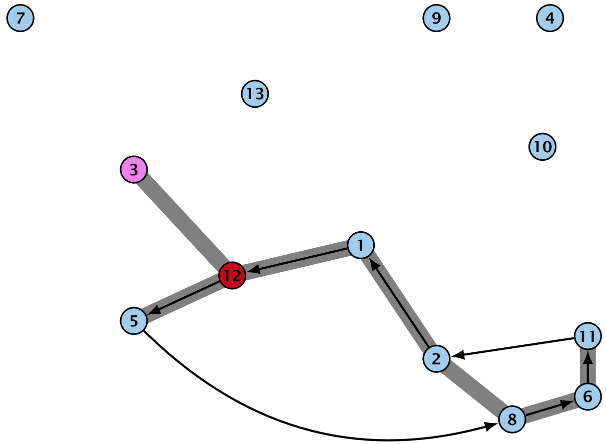

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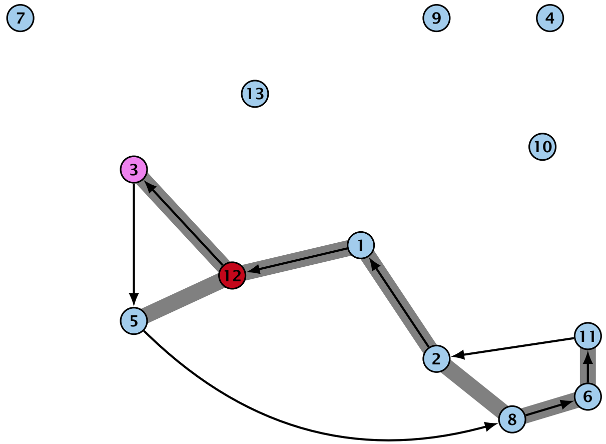

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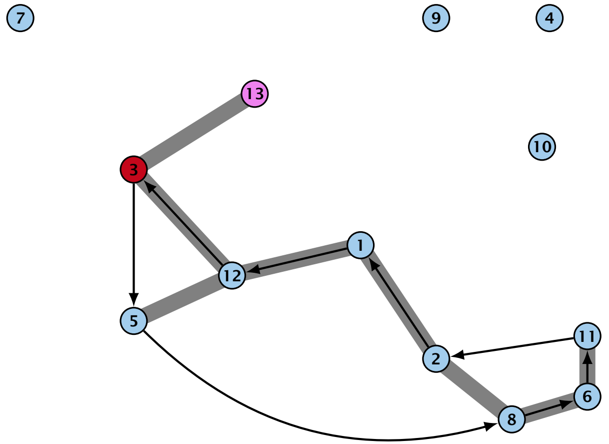

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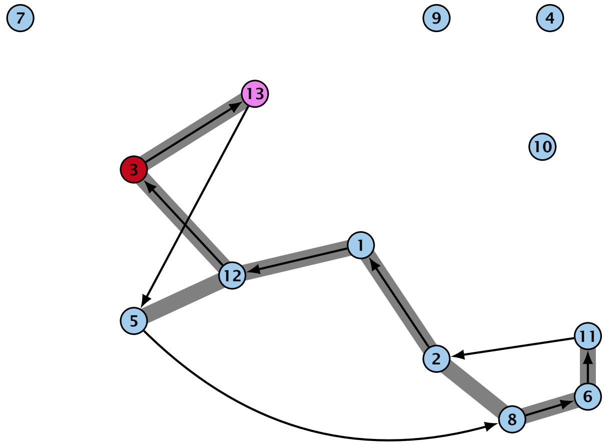

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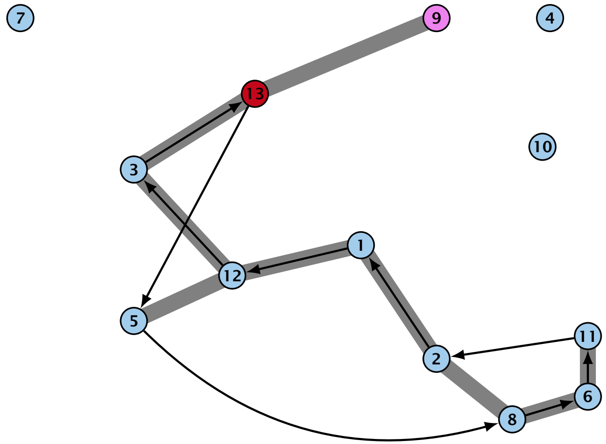

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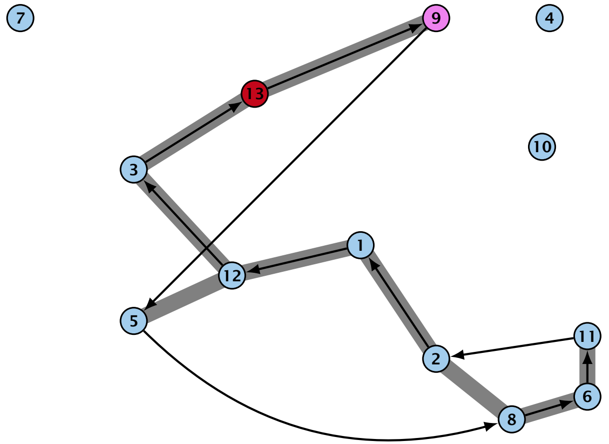

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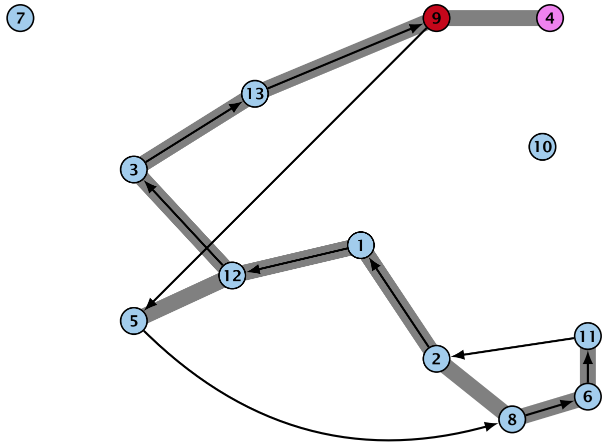

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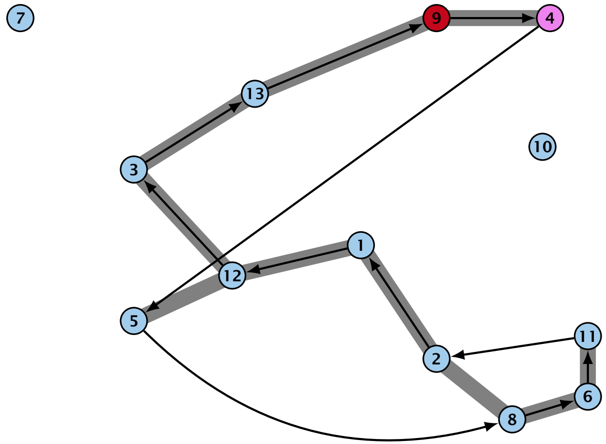

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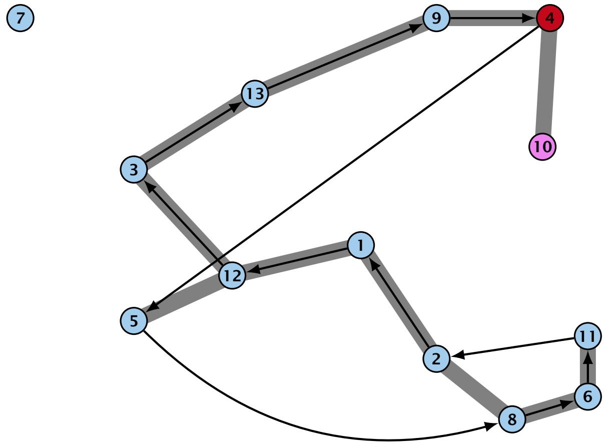

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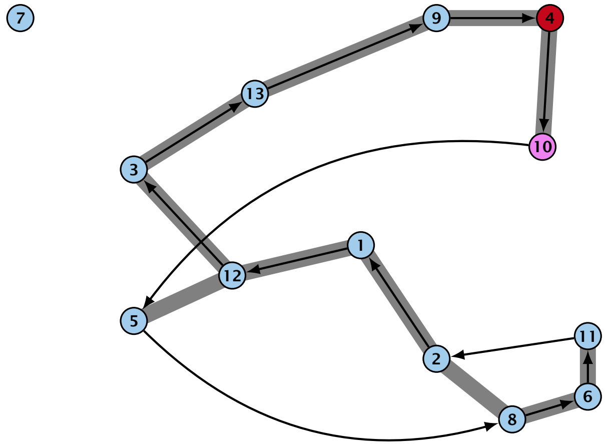

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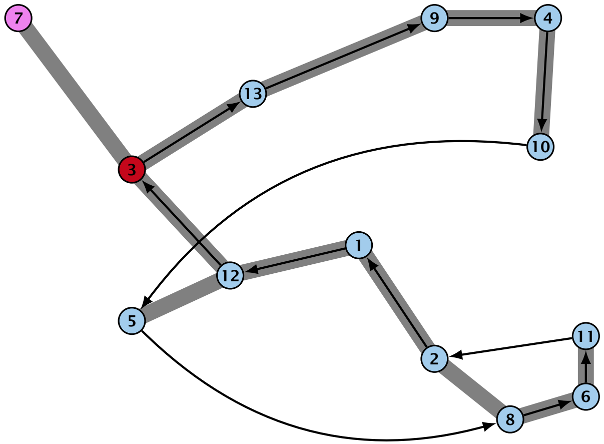

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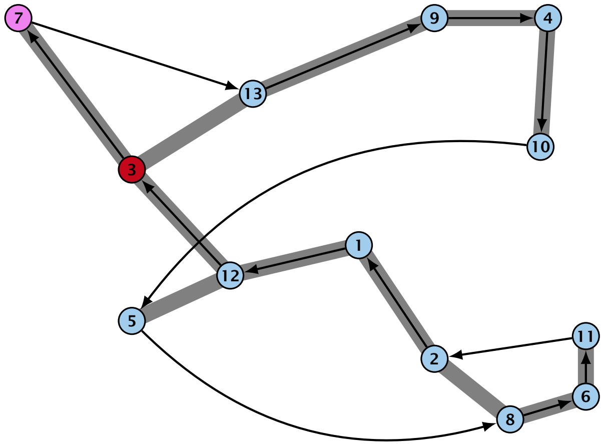

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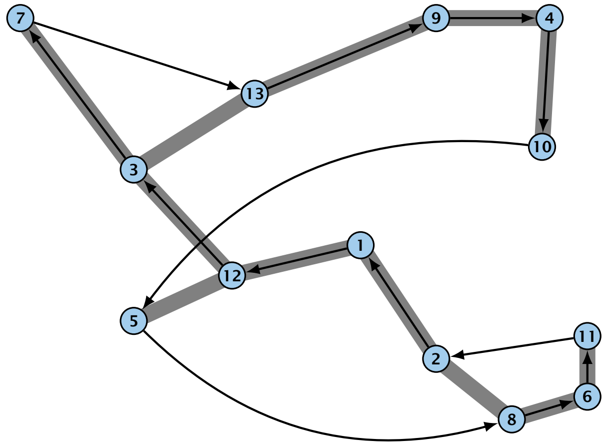

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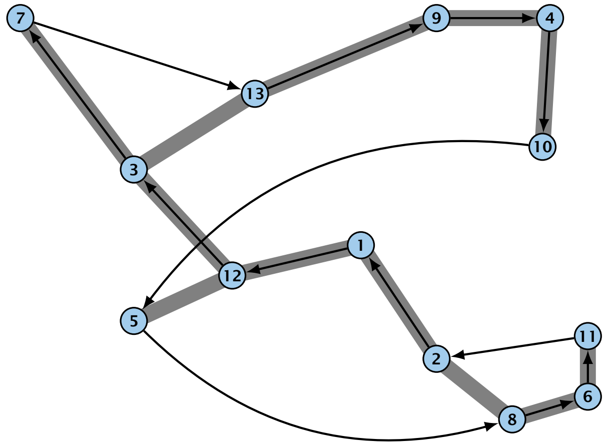

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TSP: Greedy Algorithm

Lemma 76

The Greedy algorithm is a 2*-approximation algorithm.*

Let S_i be the set at the start of the i-th iteration, and let v_i denote the node added during the iteration.

Further let $s_i \in S_i$ be the node closest to $v_i \in S_i$.

Let r_i denote the successor of s_i in the tour before inserting v_i.

We replace the edge (s_i, r_i) in the tour by the two edges (s_i, v_i) and (v_i, r_i).

This increases the cost by

$$c_{s_i,v_i} + c_{v_i,r_i} - c_{s_i,r_i} \le 2c_{s_i,v_i}$$

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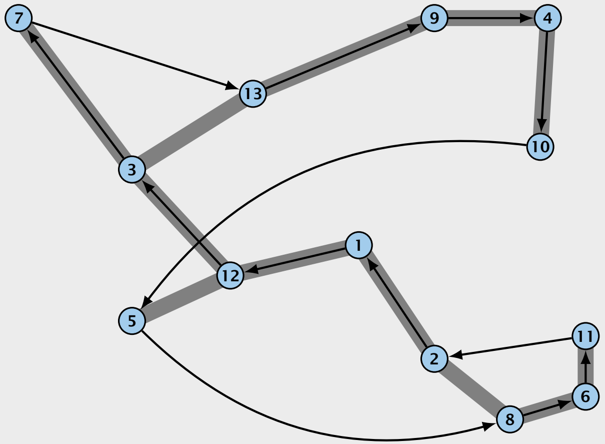

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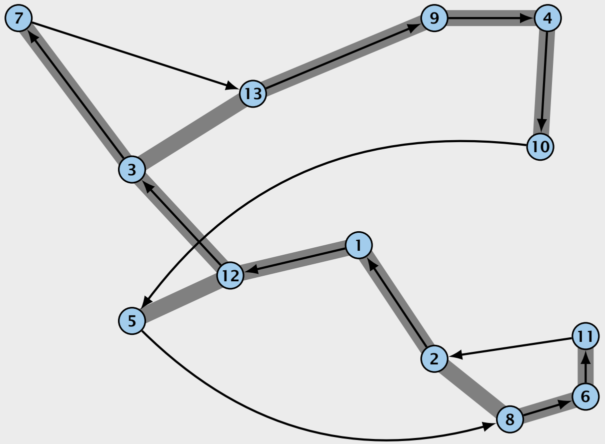

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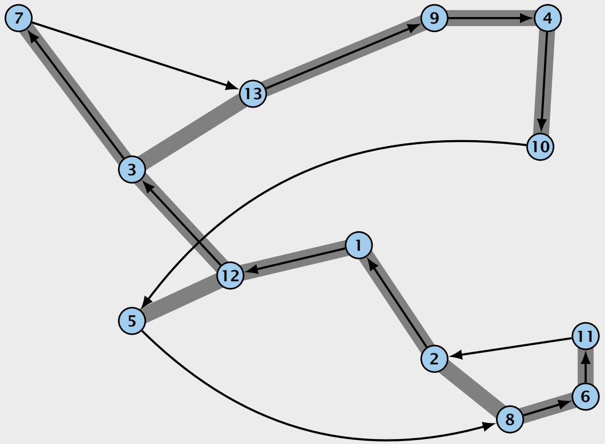

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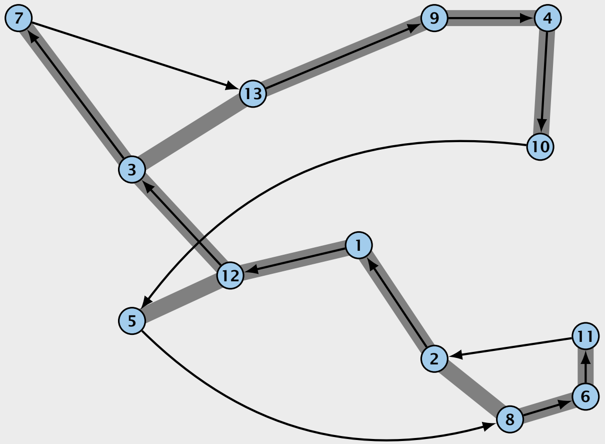

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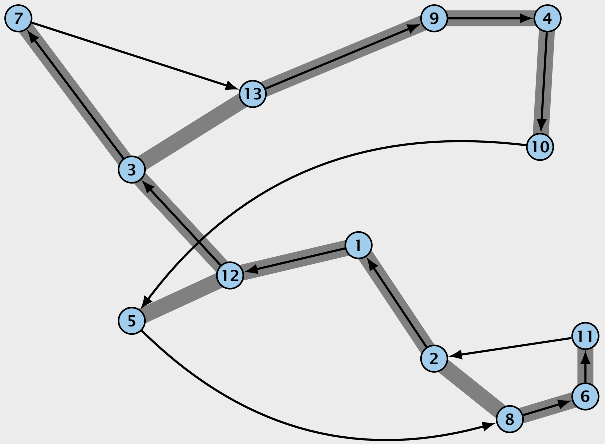

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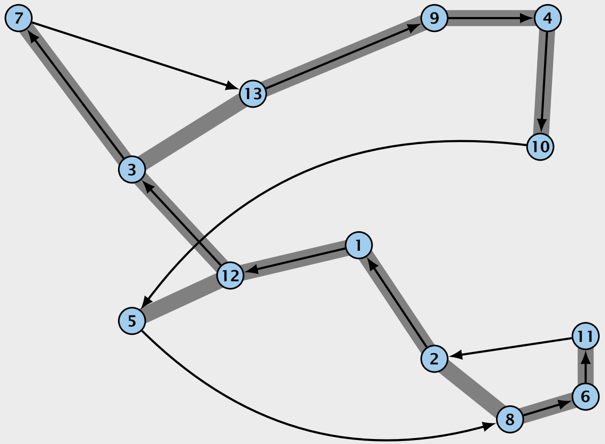

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TSP: Greedy Algorithm

The edges (s_i, v_i) considered during the Greedy algorithm are exactly the edges considered during PRIMs MST algorithm.

Hence,

$$\sum_i c_{s_i,v_i} = \mathrm{OPT}_{\mathrm{MST}}(G)$$

which with the previous lower bound gives a 2-approximation.

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which with the previous lower bound gives a 2-approximation.

TSP: Greedy Algorithm

Lemma 76
The Greedy algorithm is a 2-approximation algorithm.

Let S_i be the set at the start of the i-th iteration, and let v_i denote the node added during the iteration.

Further let $s_i \in S_i$ be the node closest to $v_i \in S_i$.

Let r_i denote the successor of s_i in the tour before inserting v_i.

We replace the edge (s_i, r_i) in the tour by the two edges (s_i, v_i) and (v_i, r_i).

This increases the cost by

$$c_{s_i,v_i} + c_{v_i,r_i} - c_{s_i,r_i} \le 2c_{s_i,v_i}$$

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Suppose that we are given an Eulerian graph $G' = (V, E', c')$ of $G = (V, E, c)$ such that for any edge $(i,j) \in E'$ $c'(i,j) \geq c(i,j)$.

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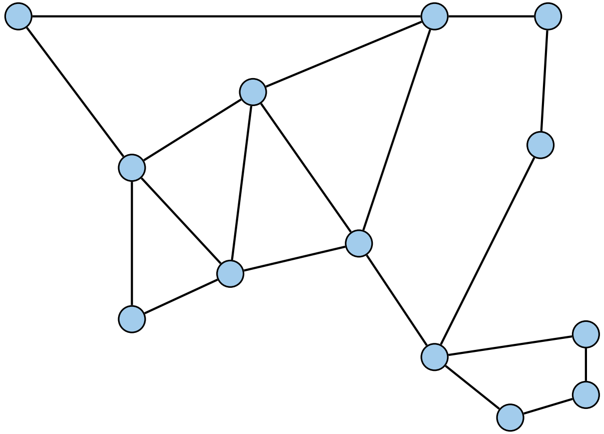

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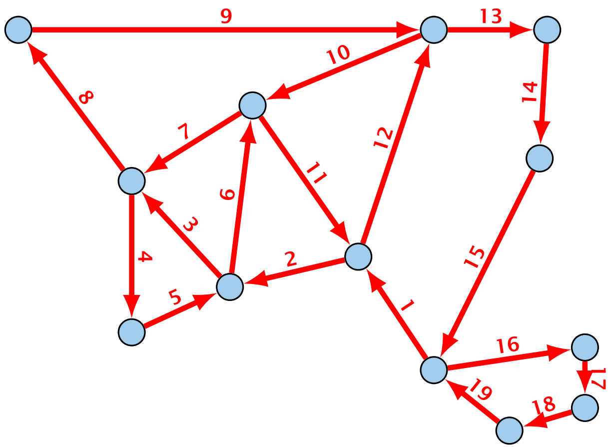

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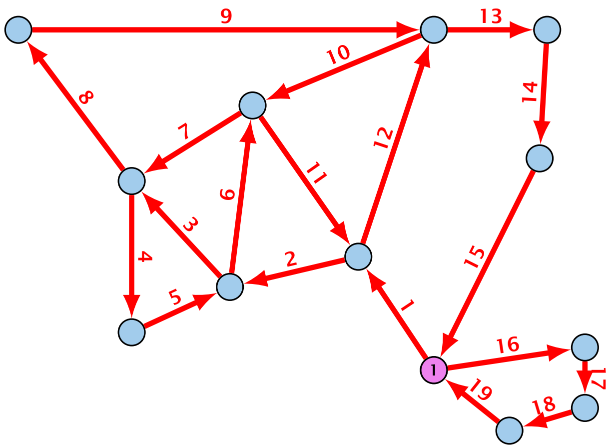

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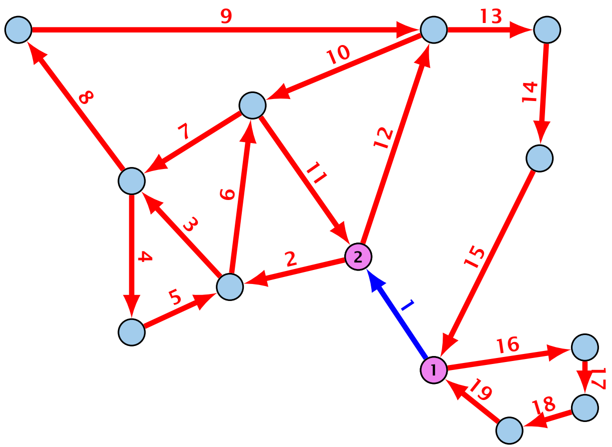

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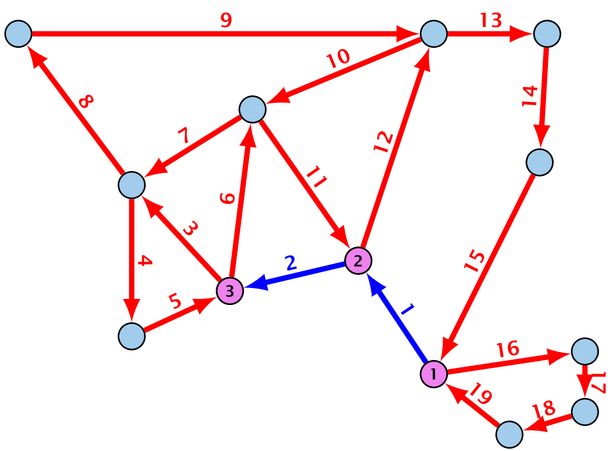

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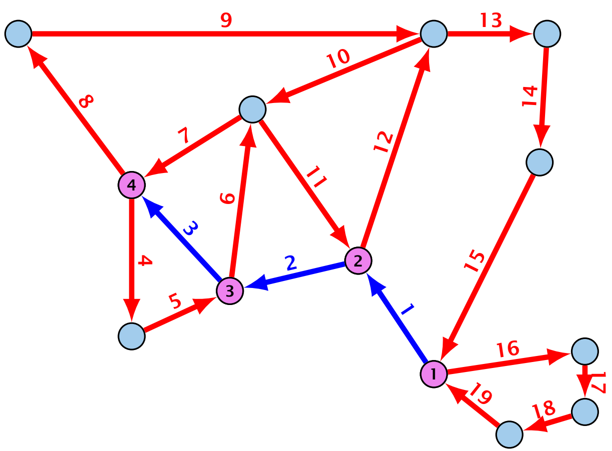

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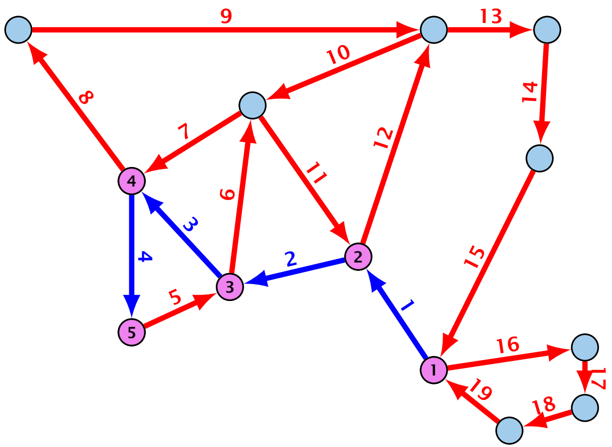

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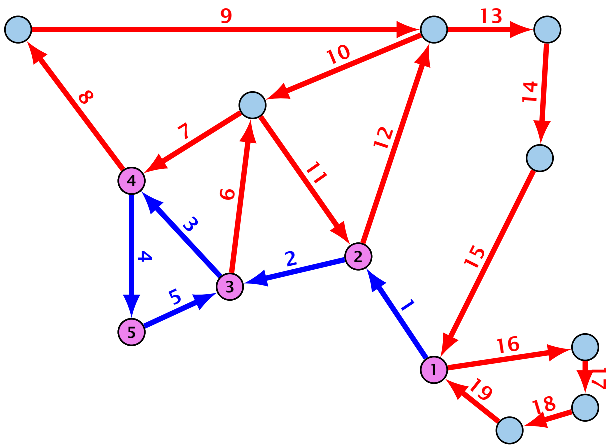

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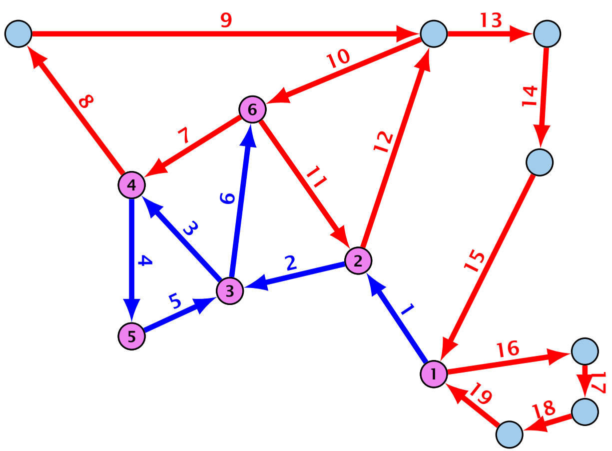

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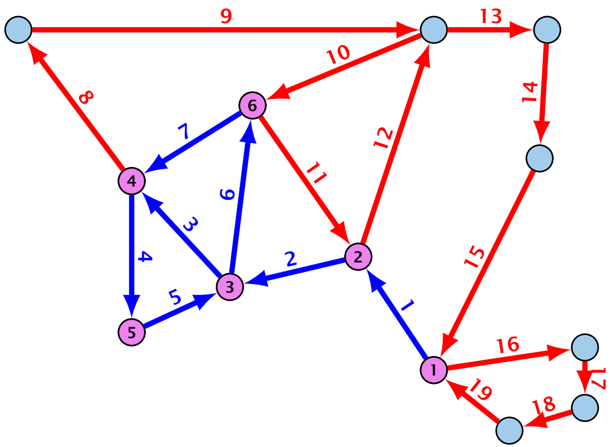

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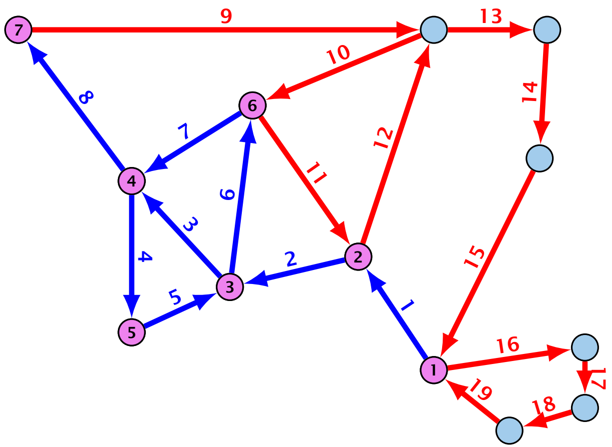

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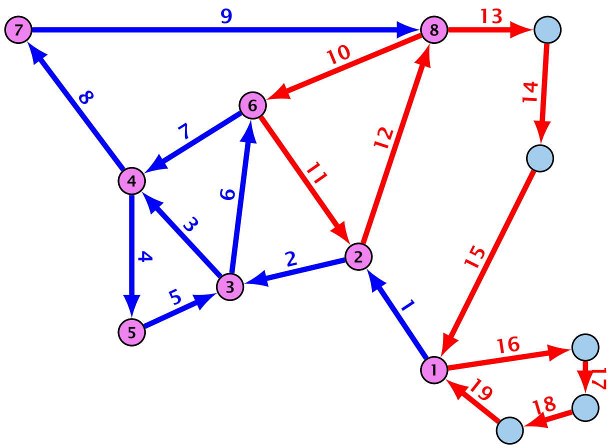

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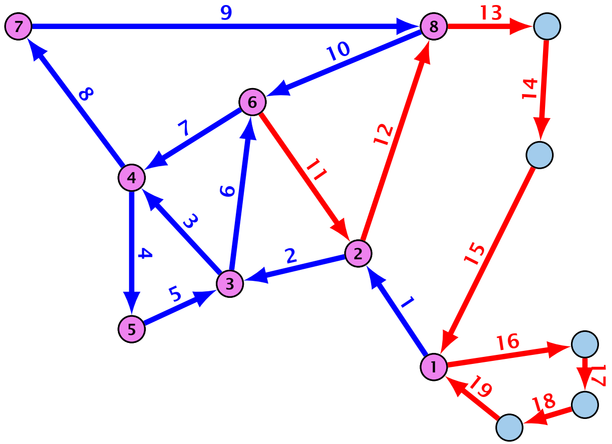

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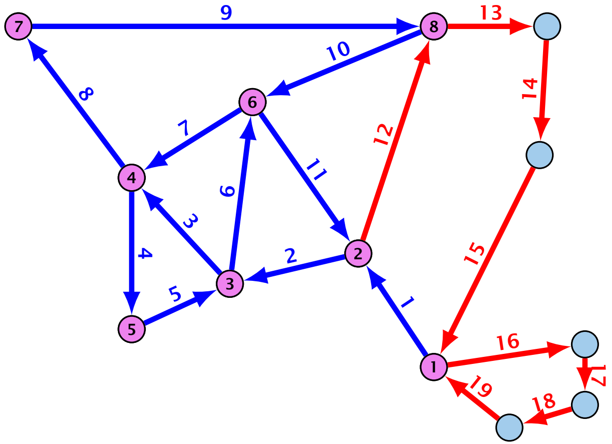

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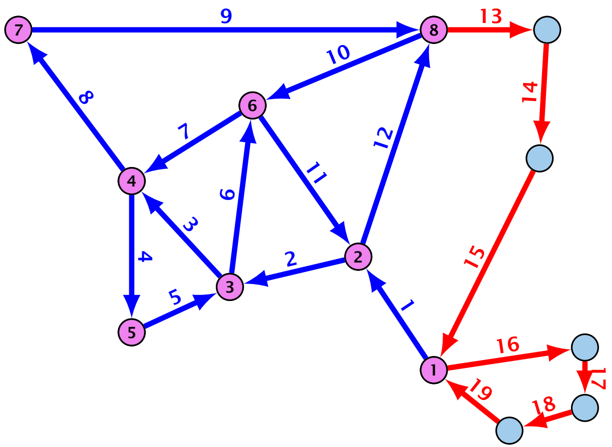

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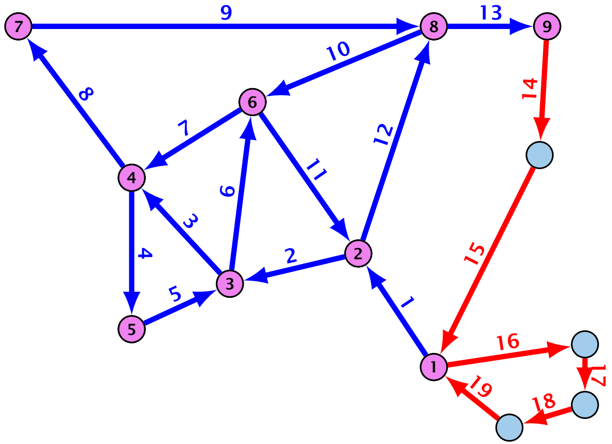

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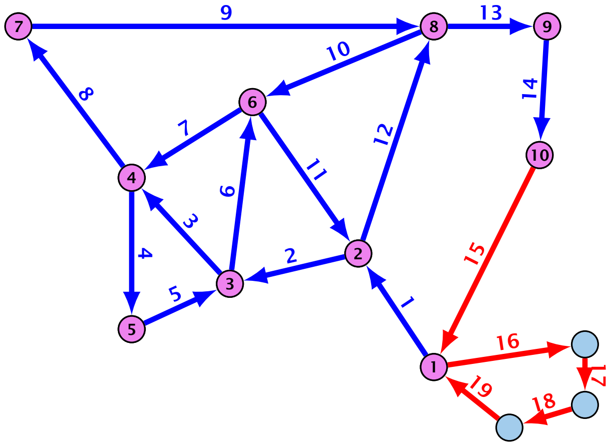

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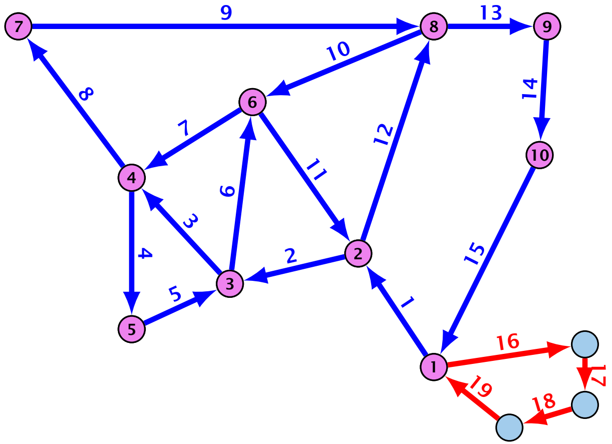

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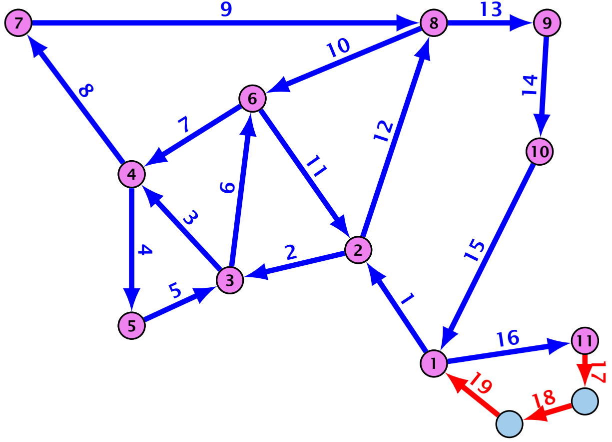

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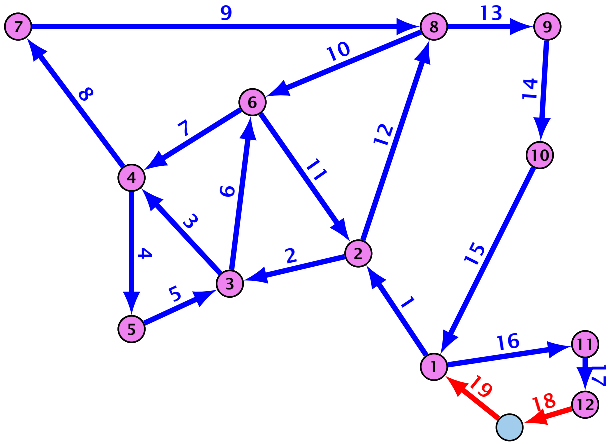

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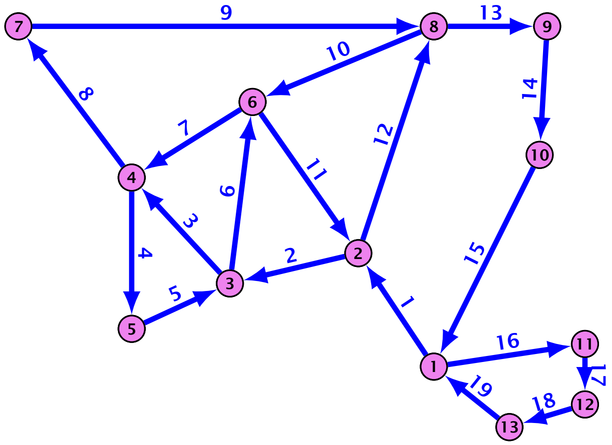

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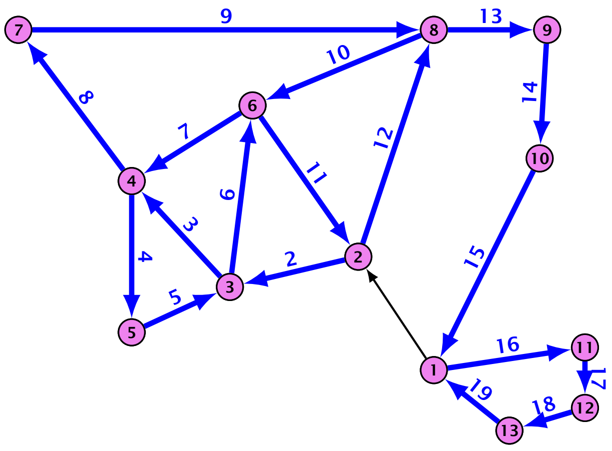

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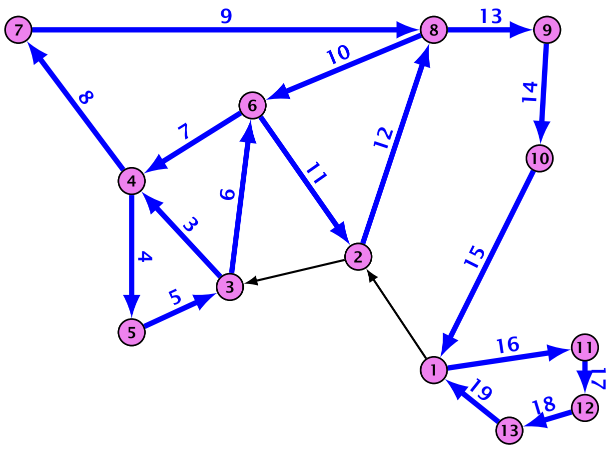

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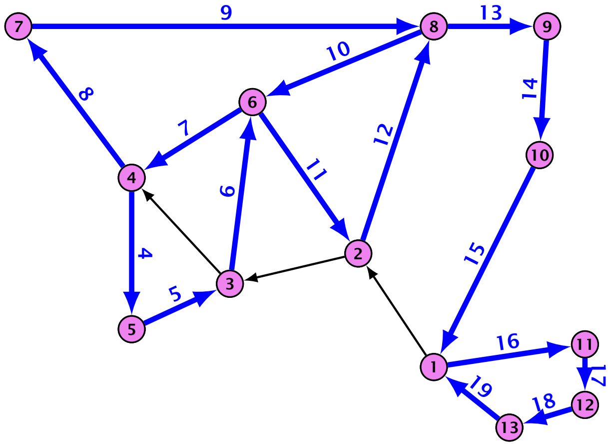

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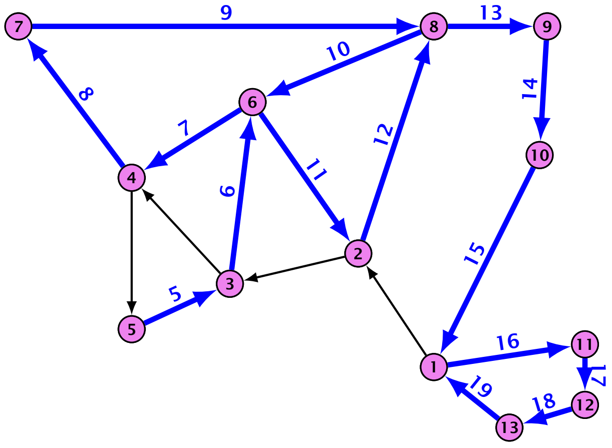

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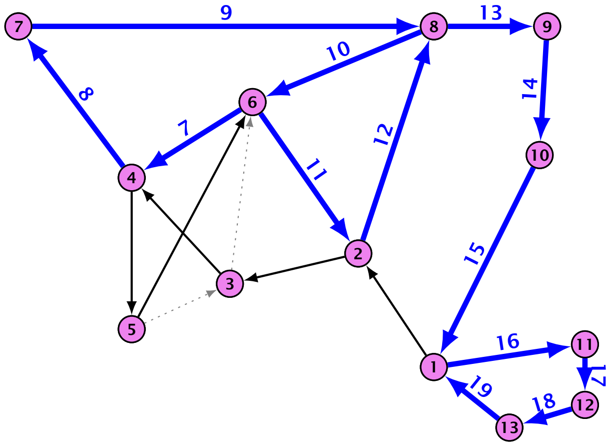

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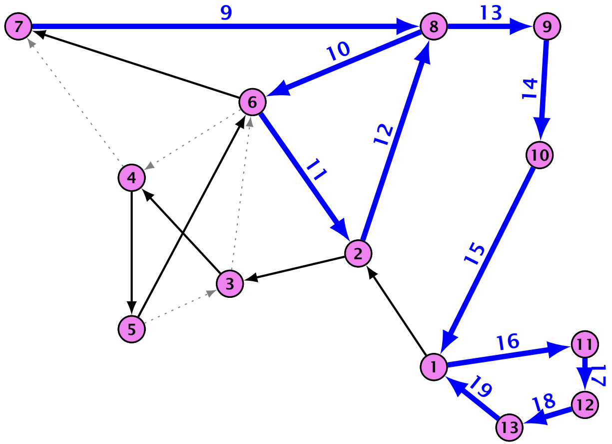

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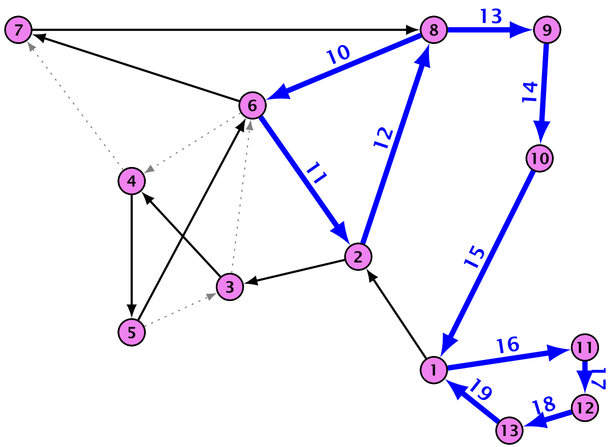

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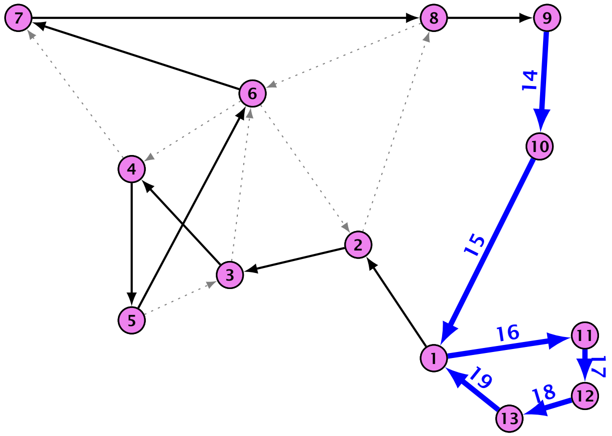

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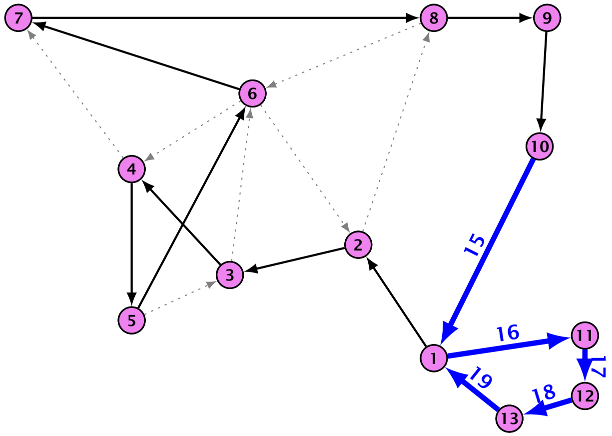

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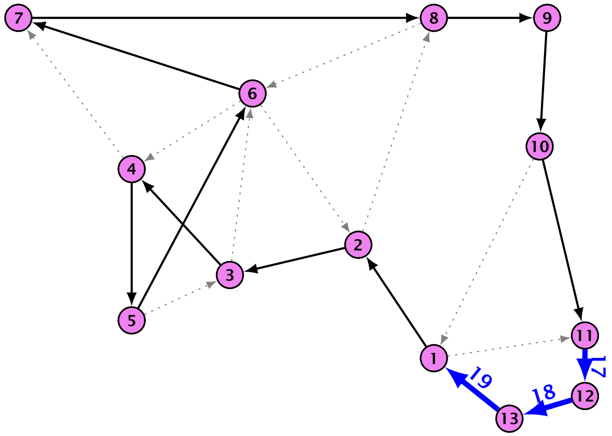

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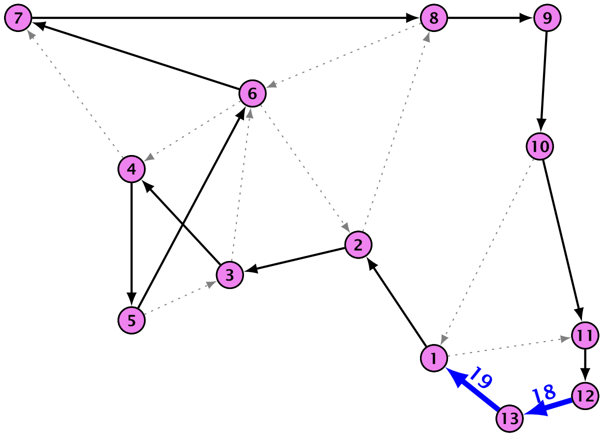

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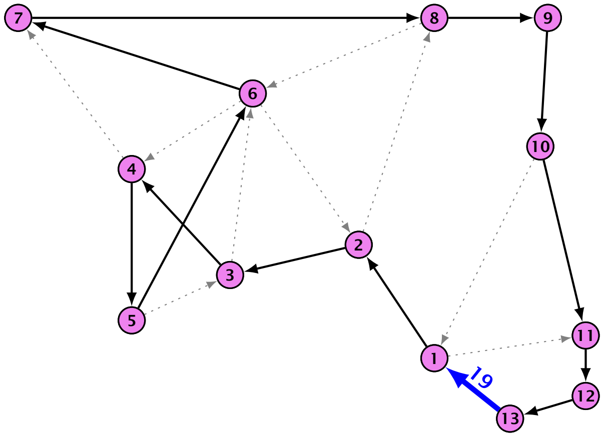

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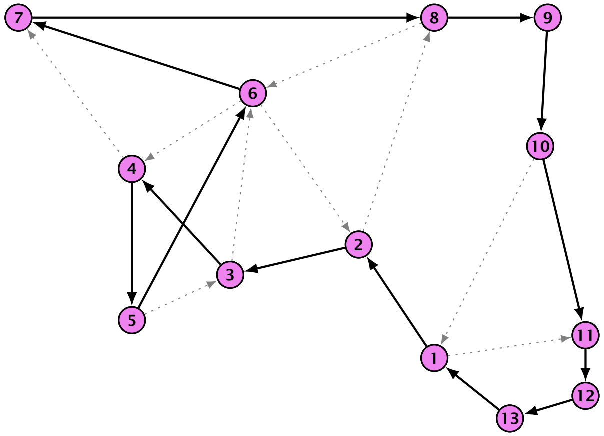

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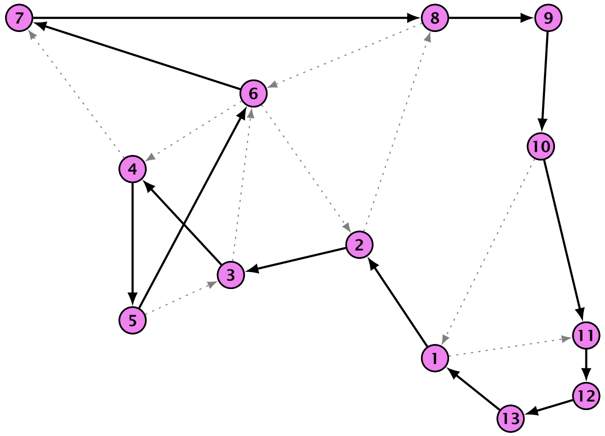

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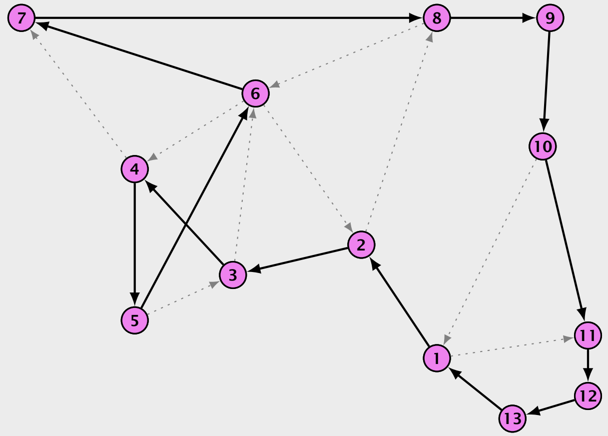

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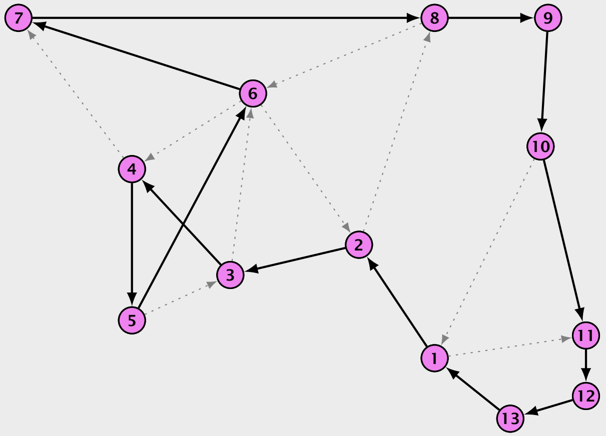

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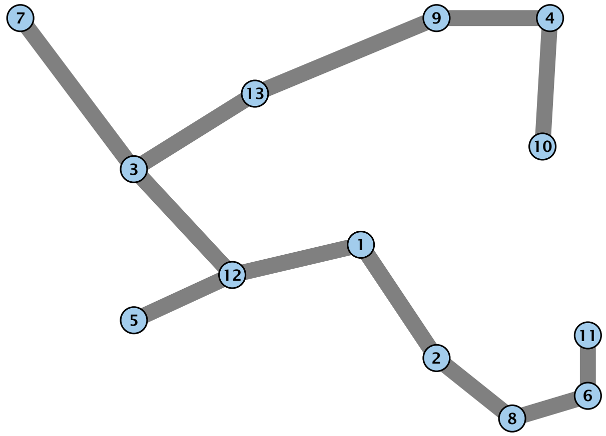

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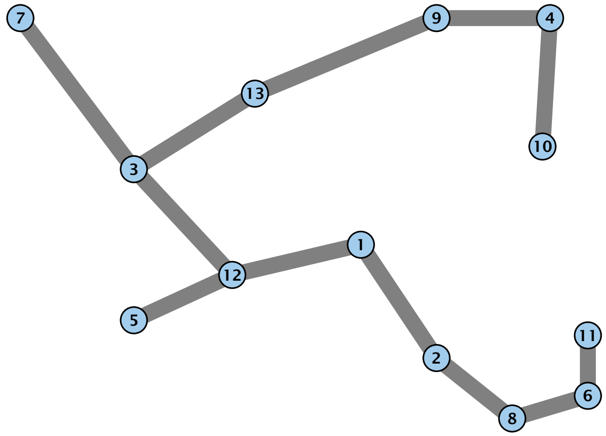

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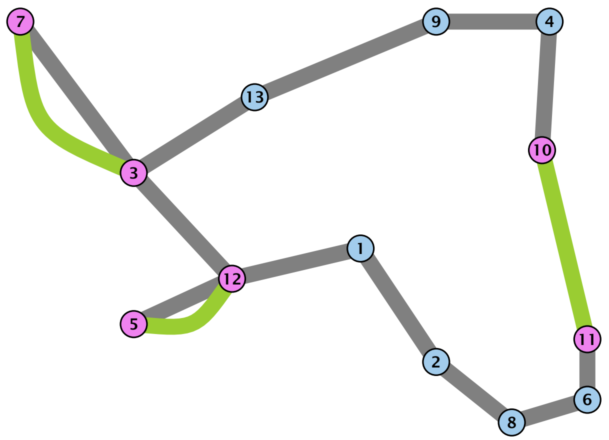

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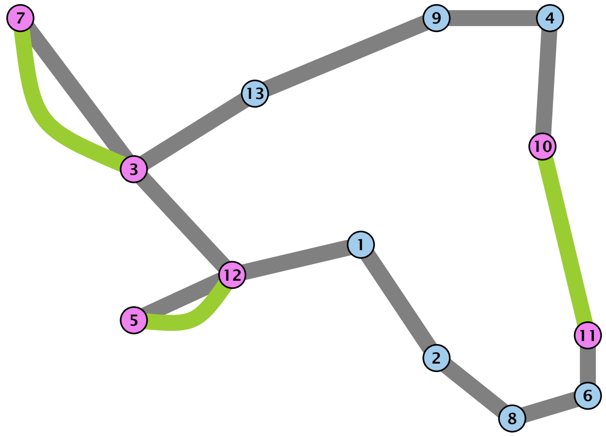

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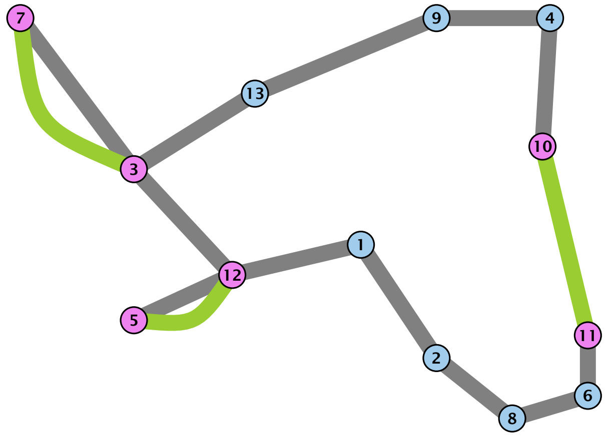

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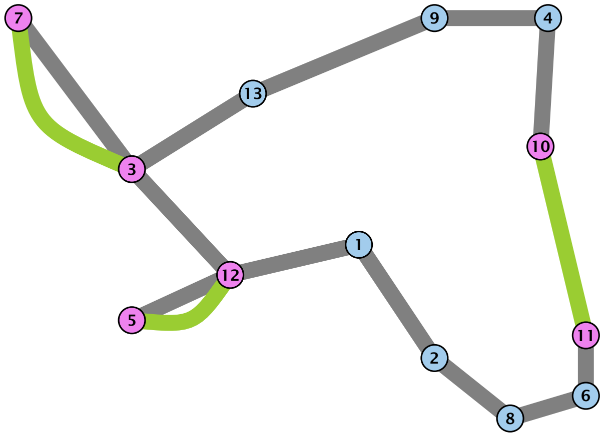

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Duplicating all edges in the MST seems to be rather wasteful.

We only need to make the graph Eulerian.

For this we compute a Minimum Weight Matching between odd degree vertices in the MST (note that there are an even number of them).

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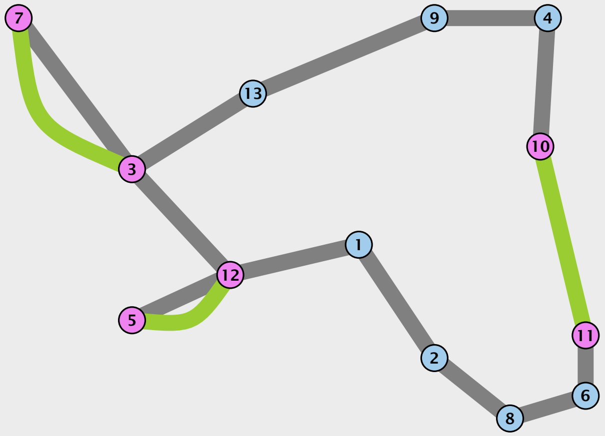

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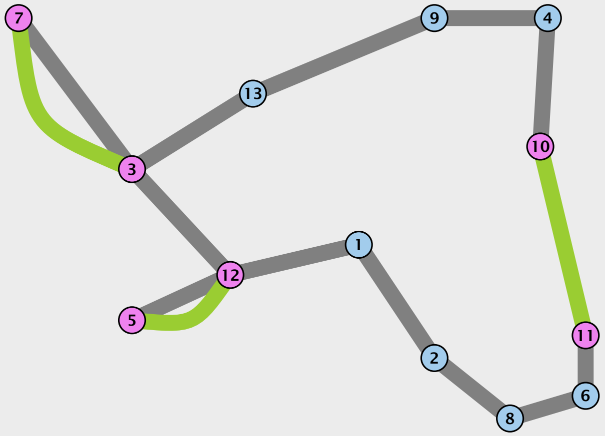

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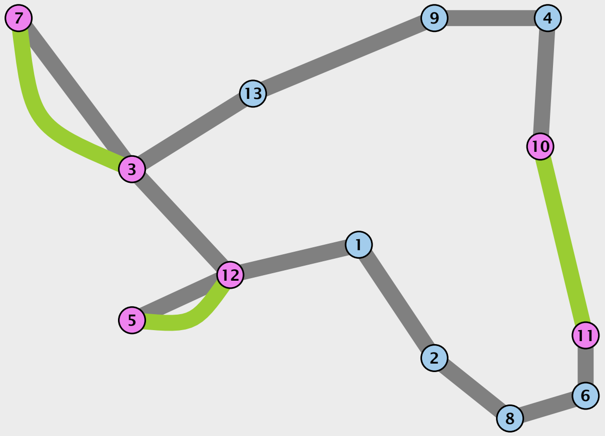

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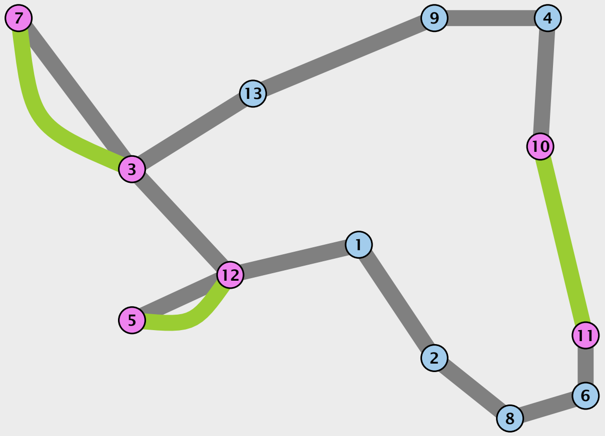

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However, the edges of this tour give rise to two disjoint matchings. One of these matchings must have weight less than $\mathrm{OPT}_{\mathrm{TSP}}(G)/2$.

Adding this matching to the MST gives an Eulerian graph with edge weight at most

$$\mathrm{OPT}_{\mathrm{MST}}(G) + \mathrm{OPT}_{\mathrm{TSP}}(G)/2 \leq \frac{3}{2}\mathrm{OPT}_{\mathrm{TSP}}(G) \ ,$$

Short cutting gives a $\frac{3}{2}$-approximation for metric TSP.

This is the best that is known.

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An optimal tour on the odd-degree vertices has cost at most $\mathrm{OPT}_{\mathrm{TSP}}(G)$.

However, the edges of this tour give rise to two disjoint matchings. One of these matchings must have weight less than $\mathrm{OPT}_{\mathrm{TSP}}(G)/2$.

Adding this matching to the MST gives an Eulerian graph with edge weight at most

$$\mathrm{OPT}_{\mathrm{MST}}(G) + \mathrm{OPT}_{\mathrm{TSP}}(G)/2 \le \frac{3}{2}\mathrm{OPT}_{\mathrm{TSP}}(G) \enspace ,$$

Short cutting gives a $\frac{3}{2}$-approximation for metric TSP.

This is the best that is known.

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Duplicating all edges in the MST seems to be rather wasteful.

We only need to make the graph Eulerian.

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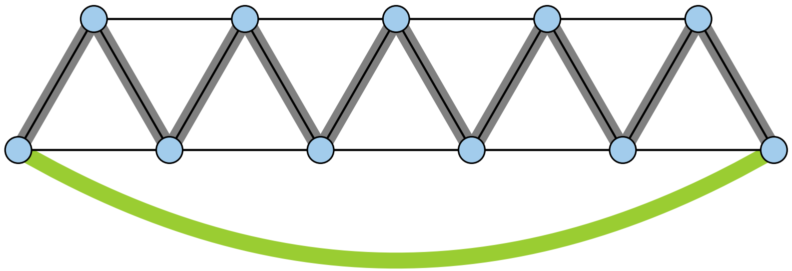
Christofides. Tight Example



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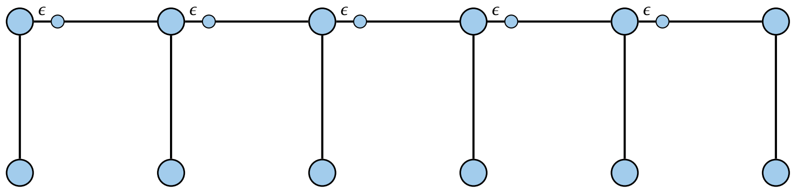

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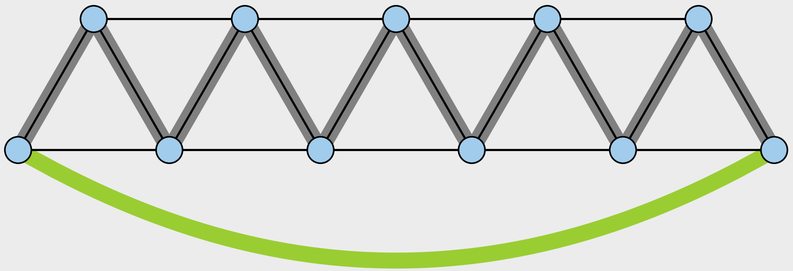

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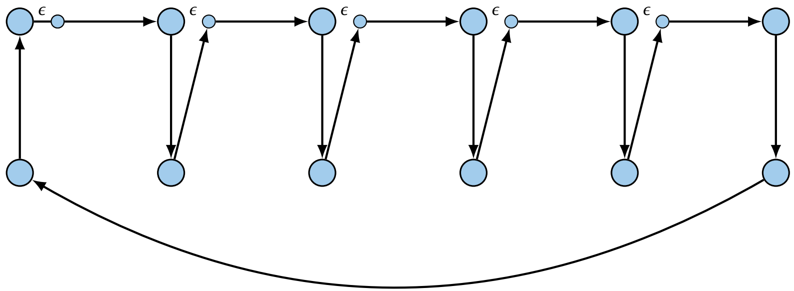

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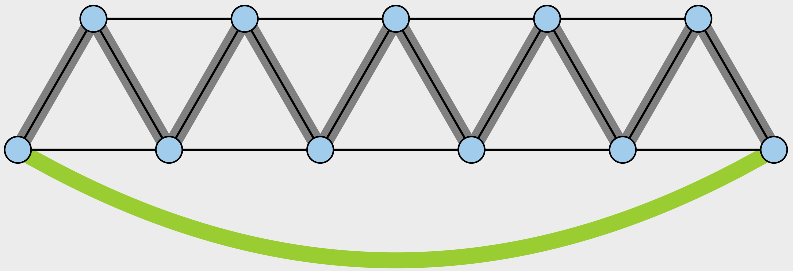

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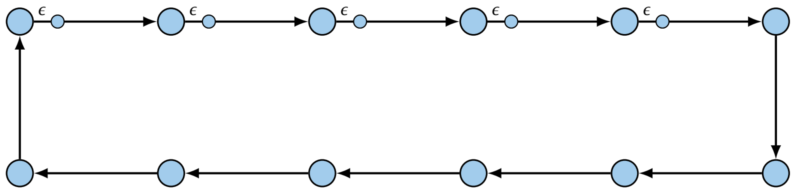

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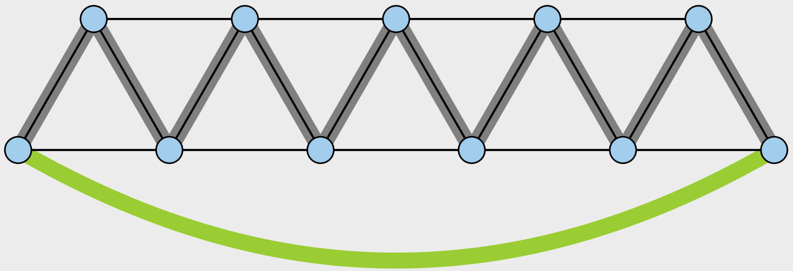

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Knapsack:

Given a set of items $\{1,\ldots,n\}$, where the i-th item has weight $w_i \in \mathbb{N}$ and profit $p_i \in \mathbb{N}$, and given a threshold W. Find a subset $I \subseteq \{1,\ldots,n\}$ of items of total weight at most W such that the profit is maximized (we can assume each $w_i \le W$).

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The running time is $\mathcal{O}(n \cdot \min\{W,P\})$, where $P = \sum_i p_i$ is the total profit of all items. This is only pseudo-polynomial.

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Partition the input into long jobs and short jobs.

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There are at most km long jobs. Hence, the number of possibilities of scheduling these jobs on m machines is at most m^{km}, which is constant if m is constant. Hence, it is easy to implement the algorithm in polynomial time.

Theorem 78
The above algorithm gives a polynomial time approximation scheme (PTAS) for the problem of scheduling n jobs on m identical machines if m is constant.

We choose $k = \lceil \frac{1}{\epsilon} \rceil$.

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How to get rid of the requirement that m is constant?

We first design an algorithm that works as follows:
On input of T it either finds a schedule of length $(1+\frac{1}{k})T$ or certifies that no schedule of length at most T exists (assume $T \geq \frac{1}{m}\sum_j p_j$).

We partition the jobs into long jobs and short jobs:

- A job is long if its size is larger than T/k.
- Otw. it is a short job.

Hence we get a schedule of length at most

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There are at most km long jobs. Hence, the number of possibilities of scheduling these jobs on m machines is at most m^{km}, which is constant if m is constant. Hence, it is easy to implement the algorithm in polynomial time.

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The above algorithm gives a polynomial time approximation scheme (PTAS) for the problem of scheduling n jobs on m identical machines if m is constant.

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- We round all long jobs down to multiples of T/k^2.
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After the first phase the rounded sizes of the long jobs assigned to a machine add up to at most T.

There can be at most k (long) jobs assigned to a machine as otw. their rounded sizes would add up to more than T (note that the rounded size of a long job is at least T/k).

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During the second phase there always must exist a machine with load at most T, since T is larger than the average load.
Assigning the current (short) job to such a machine gives that the new load is at most

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Running Time for scheduling large jobs: There should not be a job with rounded size more than T as otw. the problem becomes trivial.

Hence, any large job has rounded size of $\frac{i}{k^2}T$ for $i \in \{k, \dots, k^2\}$. Therefore the number of different inputs is at most n^{k^2} (described by a vector of length k^2 where, the i-th entry describes the number of jobs of size $\frac{i}{k^2}T$). This is polynomial.

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We can turn this into a PTAS by choosing $k = \lceil 1/\epsilon \rceil$ and using binary search. This gives a running time that is exponential in $1/\epsilon$.

Can we do better?
Scheduling on identical machines with the goal of minimizing Makespan is a strongly NP-complete problem.

Theorem 79
There is no FPTAS for problems that are strongly NP-hard.

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- Then

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- But this means that the algorithm computes the optimal solution as the optimum is integral.
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- For strongly NP-complete problems this is not possible unless P=NP

We can turn this into a PTAS by choosing $k = \lceil 1/\epsilon \rceil$ and using binary search. This gives a running time that is exponential in $1/\epsilon$.

Can we do better?
Scheduling on identical machines with the goal of minimizing Makespan is a strongly NP-complete problem.

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There is no FPTAS for problems that are strongly NP-hard.

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More General

Let $\mathrm{OPT}(n_1, \dots, n_A)$ be the number of machines that are required to schedule input vector $(n_1, \dots, n_A)$ with Makespan at most T (A: number of different sizes).

If $\mathrm{OPT}(n_1, \dots, n_A) \le m$ we can schedule the input.

$$\mathrm{OPT}(n_1, \dots, n_A) = \begin{cases} 0 & (n_1, \dots, n_A) = 0 \\ 1 + \min\limits_{(s_1, \dots, s_A) \in C} \mathrm{OPT}(n_1 - s_1, \dots, n_A - s_A) & (n_1, \dots, n_A) \gneq 0 \\ \infty & \text{otw.} \end{cases}$$

where C is the set of all configurations.

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Bin Packing

Given n items with sizes $s_1, \ldots, s_n$ where

$$1 > s_1 \geq \cdots \geq s_n > 0 \ .$$

Pack items into a minimum number of bins where each bin can hold items of total size at most 1.

Theorem 80

There is no ρ-approximation for Bin Packing with $\rho < 3/2$ unless $P = NP$.

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Proof

- In the partition problem we are given positive integers $b_1, \ldots, b_n$ with $B = \sum_i b_i$ even. Can we partition the integers into two sets S and T s.t.

$$\sum_{i \in S} b_i = \sum_{i \in T} b_i \quad ?$$

- We can solve this problem by setting $s_i := 2b_i/B$ and asking whether we can pack the resulting items into 2 bins or not.
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Definition 81
An asymptotic polynomial-time approximation scheme (APTAS) is a family of algorithms $\{A_\epsilon\}$ along with a constant c such that A_ϵ returns a solution of value at most $(1+\epsilon)\mathrm{OPT} + c$ for minimization problems.

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Again we can differentiate between small and large items.

Lemma 82

Any packing of items into ℓ bins can be extended with items of size at most γ s.t. we use only $\max\{\ell, \frac{1}{1-\gamma}\mathrm{SIZE}(I) + 1\}$ bins, where $\mathrm{SIZE}(I) = \sum_i s_i$ is the sum of all item sizes.

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- If after Greedy we use more than ℓ bins, all bins (apart from the last) must be full to at least $1 - \gamma$.
- Hence, $r(1-\gamma) \le \mathrm{SIZE}(I)$ where r is the number of nearly-full bins.
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Bin Packing

Linear Grouping:

Generate an instance I' (for large items) as follows.

- Order large items according to size.
- Let the first k items belong to group 1; the following k items belong to group 2; etc.
- Delete items in the first group;
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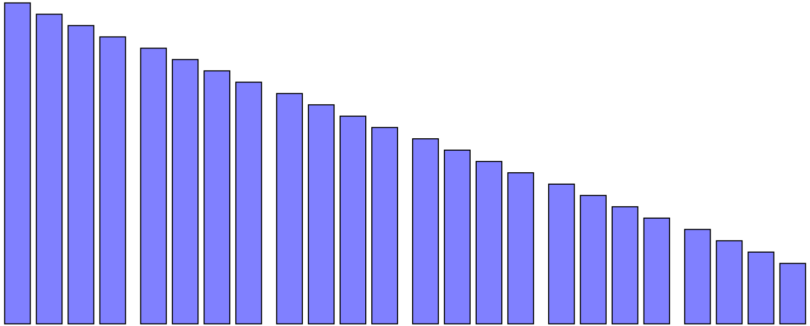

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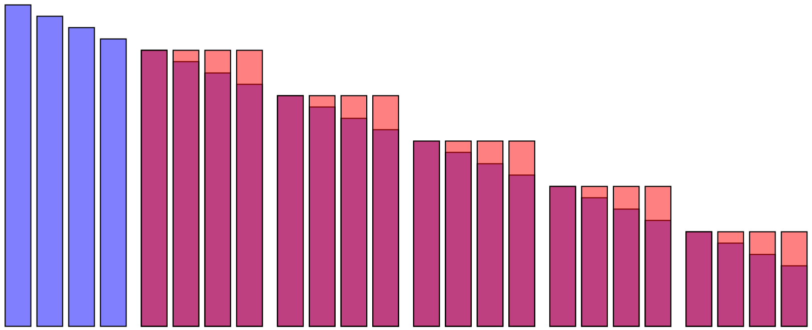

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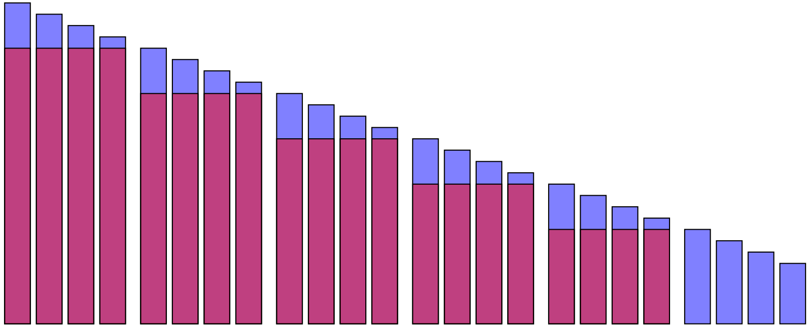

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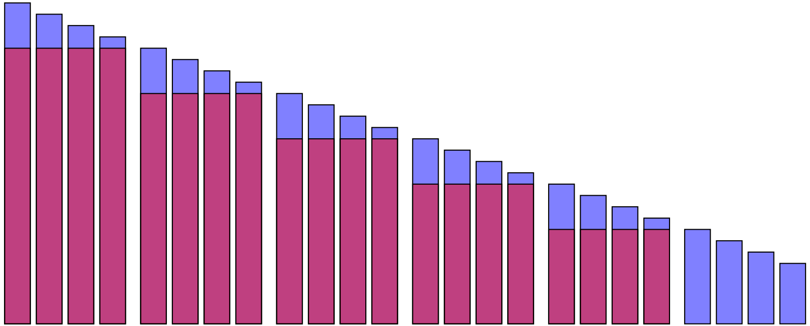

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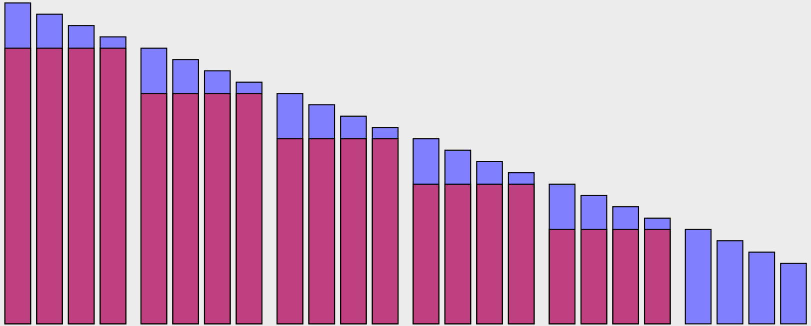
Lemma 83

$\mathrm{OPT}(I') \le \mathrm{OPT}(I) \le \mathrm{OPT}(I') + k$

Proof 1:

- Any bin packing for I gives a bin packing for I' as follows.
- Pack the items of group 2, where in the packing for I the items for group 1 have been packed.
- Pack the items of groups 3, where in the packing for I the items for group 2 have been packed.
- ...

Linear Grouping



Lemma 83

$\mathrm{OPT}(I') \leq \mathrm{OPT}(I) \leq \mathrm{OPT}(I') + k$

Proof 1:

- Any bin packing for I gives a bin packing for I' as follows.
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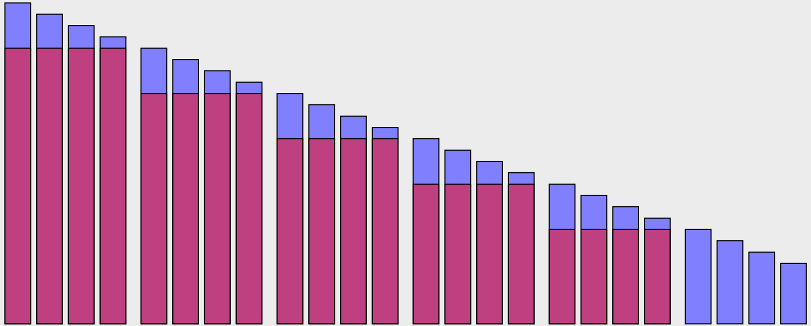
Linear Grouping



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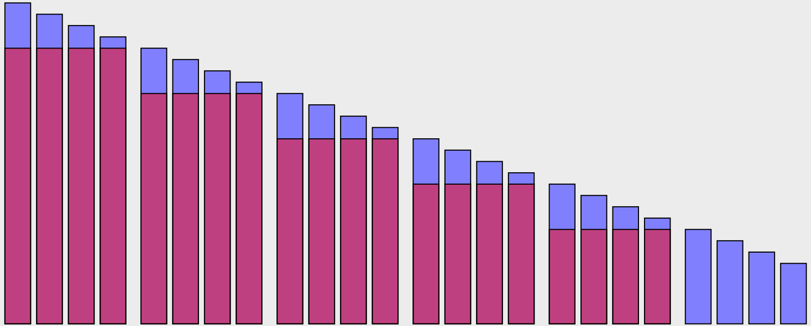

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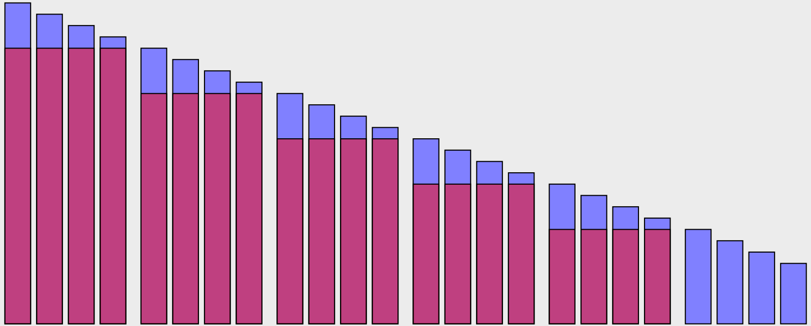

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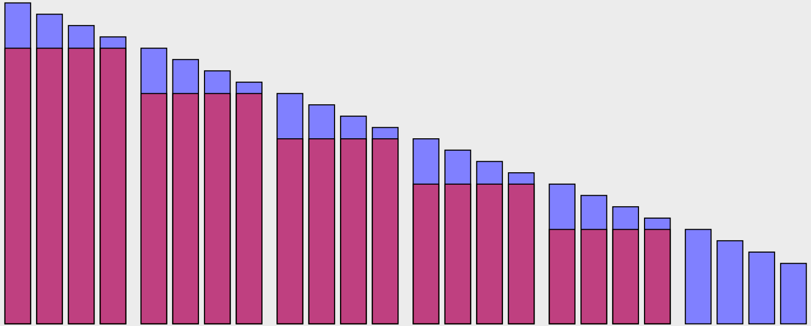

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We can find an optimal packing for such instances by the previous Dynamic Programming approach.

- cost (for large items) at most

$$\mathrm{OPT}(I') + k \leq \mathrm{OPT}(I) + \epsilon \mathrm{SIZE}(I) \leq (1+\epsilon)\mathrm{OPT}(I)$$

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Change of Notation:

- Group pieces of identical size.
- Let s_1 denote the largest size, and let b_1 denote the number of pieces of size s_1.
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A possible packing of a bin can be described by an m-tuple $(t_1, \dots, t_m)$, where t_i describes the number of pieces of size s_i. Clearly,

$$\sum_i t_i \cdot s_i \le 1 \ .$$

We call a vector that fulfills the above constraint a configuration.

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Let N be the number of configurations (exponential).

Let $T_1, \dots, T_N$ be the sequence of all possible configurations (a configuration T_j has T_{ji} pieces of size s_i).

$$\begin{array}{llrcl} \min & & \sum_{j=1}^{N} x_j & & \\ \text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\ & \forall j \in \{1, \dots, N\} & x_j & \geq & 0 \\ & \forall j \in \{1, \dots, N\} & x_j & \text{integral} & \end{array}$$

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- Sort items according to size (monotonically decreasing).
- Process items in this order; close the current group if size of items in the group is at least 2 (or larger). Then open new group.
- I.e., G_1 is the smallest cardinality set of largest items s.t. total size sums up to at least 2. Similarly, for $G_2, \dots, G_{r-1}$.
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- I.e., G_1 is the smallest cardinality set of largest items s.t. total size sums up to at least 2. Similarly, for $G_2, \ldots, G_{r-1}$.
- Only the size of items in the last group G_r may sum up to less than 2.

We can assume that each item has size at least $1/\mathrm{SIZE}(I)$.

Harmonic Grouping

From the grouping we obtain instance I' as follows:

- Round all items in a group to the size of the largest group member.
- Delete all items from group G_1 and G_r.
- For groups $G_2, \dots, G_{r-1}$ delete $n_i - n_{i-1}$ items.
- Observe that $n_i \geq n_{i-1}$.

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Lemma 85
The number of different sizes in I' is at most $\mathrm{SIZE}(I)/2$.

- Each group that survives (recall that G_1 and G_r are deleted) has total size at least 2.
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Lemma 86
The total size of deleted items is at most $\mathcal{O}(\log(\text{SIZE}(I)))$.

- The total size of items in G_1 and G_r is at most 6 as a group has total size at most 3.
- Consider a group G_i that has strictly more items than G_{i-1}.
- It discards $n_i - n_{i-1}$ pieces of total size at most

$$3\frac{n_i - n_{i-1}}{n_i} \le \sum_{j=n_{i-1}+1}^{n_i} \frac{3}{j}$$

  since the smallest piece has size at most $3/n_i$.
- Summing over all i that have $n_i > n_{i-1}$ gives a bound of at most

$$\sum_{j=1}^{n_{r-1}} \frac{3}{j} \le \mathcal{O}(\log(\text{SIZE}(I))) \ .$$

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Algorithm 1 BinPack

if $\mathrm{SIZE}(I) < 10$ **then**
    pack remaining items greedily
Apply harmonic grouping to create instance I'; pack discarded items in at most $\mathcal{O}(\log(\mathrm{SIZE}(I)))$ bins.
Let x be optimal solution to configuration LP
Pack $\lfloor x_j \rfloor$ bins in configuration T_j for all j; call the packed instance I_1.
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Pack I_2 via BinPack(I_2)

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$$\mathrm{OPT_{LP}}(I_1) + \mathrm{OPT_{LP}}(I_2) \le \mathrm{OPT_{LP}}(I') \le \mathrm{OPT_{LP}}(I)$$

Proof:

- Each piece surviving in I' can be mapped to a piece in I of no lesser size. Hence, $\mathrm{OPT_{LP}}(I') \le \mathrm{OPT_{LP}}(I)$
- $\lfloor x_j \rfloor$ is feasible solution for I_1 (even integral).
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EADS II
Harald Räcke
16.4 Advanced Rounding for Bin Packing

Analysis

Each level of the recursion partitions pieces into three types

1. Pieces discarded at this level.
2. Pieces scheduled because they are in I_1.
3. Pieces in I_2 are handed down to the next level.

Pieces of type 2 summed over all recursion levels are packed into at most $\mathrm{OPT}_{\mathrm{LP}}$ many bins.

Pieces of type 1 are packed into at most

$$\mathcal{O}(\log(\mathrm{SIZE}(I))) \cdot L$$

many bins where L is the number of recursion levels.

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How to solve the LP?

Let $T_1, \dots, T_N$ be the sequence of all possible configurations (a configuration T_j has T_{ji} pieces of size s_i).
In total we have b_i pieces of size s_i.

Primal

$$\begin{array}{llcl} \min & & \sum_{j=1}^{N} x_j & \\ \text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\ & \forall j \in \{1, \dots, N\} & x_j & \geq & 0 \end{array}$$

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$$\begin{array}{llcl} \max & & \sum_{i=1}^{m} y_i b_i & \\ \text{s.t.} & \forall j \in \{1, \dots, N\} & \sum_{i=1}^{m} T_{ji} y_i & \leq & 1 \\ & \forall i \in \{1, \dots, m\} & y_i & \geq & 0 \end{array}$$

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$$\begin{array}{llrcl} \min & & \sum_{j=1}^{N} x_j & & \\ \text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\ & \forall j \in \{1, \dots, N\} & x_j & \geq & 0 \end{array}$$

Dual

$$\begin{array}{llrcl} \max & & \sum_{i=1}^{m} y_i b_i & & \\ \text{s.t.} & \forall j \in \{1, \dots, N\} & \sum_{i=1}^{m} T_{ji} y_i & \leq & 1 \\ & \forall i \in \{1, \dots, m\} & y_i & \geq & 0 \end{array}$$

Analysis

We can show that $\text{SIZE}(I_2) \leq \text{SIZE}(I)/2$. Hence, the number of recursion levels is only $\mathcal{O}(\log(\text{SIZE}(I_{\text{original}})))$ in total.

- The number of non-zero entries in the solution to the configuration LP for I' is at most the number of constraints, which is the number of different sizes ($\leq \text{SIZE}(I)/2$).
- The total size of items in I_2 can be at most $\sum_{j=1}^{N} x_j - \lfloor x_j \rfloor$ which is at most the number of non-zero entries in the solution to the configuration LP.

Separation Oracle

Suppose that I am given variable assignment y for the dual.

How do I find a violated constraint?

I have to find a configuration $T_j = (T_{j1}, \dots, T_{jm})$ that

- is feasible, i.e.,

$$\sum_{i=1}^{m} T_{ji} \cdot s_i \le 1 \;,$$

- and has a large profit

$$\sum_{i=1}^{m} T_{ji} y_i > 1$$

But this is the Knapsack problem.

How to solve the LP?

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We have FPTAS for Knapsack. This means if a constraint is violated with $1 + \epsilon' = 1 + \frac{\epsilon}{1-\epsilon}$ we find it, since we can obtain at least $(1 - \epsilon)$ of the optimal profit.

The solution we get is feasible for:

Dual'

$$\begin{array}{llrcl} \max & & \sum_{i=1}^{m} y_i b_i & & \\ \text{s.t.} & \forall j \in \{1,\dots,N\} & \sum_{i=1}^{m} T_{ji} y_i & \le & 1+\epsilon' \\ & \forall i \in \{1,\dots,m\} & y_i & \ge & 0 \end{array}$$

Primal'

$$\begin{array}{llrcl} \min & & (1+\epsilon') \sum_{j=1}^{N} x_j & & \\ \text{s.t.} & \forall i \in \{1\dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \ge & b_i \\ & \forall j \in \{1,\dots,N\} & x_j & \ge & 0 \end{array}$$

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If the value of the computed dual solution (which may be infeasible) is z then

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This gives that overall we need at most

$$(1+\epsilon')\mathrm{OPT}_{\mathrm{LP}}(I) + \mathcal{O}(\log^2(\mathrm{SIZE}(I)))$$

bins.

We can choose $\epsilon' = \frac{1}{\mathrm{OPT}}$ as $\mathrm{OPT} \le \#\text{items}$ and since we have a fully polynomial time approximation scheme (FPTAS) for knapsack.

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Lemma 87 (Chernoff Bounds)

Let $X_1, \ldots, X_n$ be n independent 0-1 random variables, not necessarily identically distributed. Then for $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$, $L \le \mu \le U$, and $\delta > 0$

$$\Pr[X \geq (1+\delta)U] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \quad ,$$

and

$$\Pr[X \leq (1-\delta)L] < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} \quad ,$$

Lemma 88

For $0 \leq \delta \leq 1$ *we have that*

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta^2/3}$$

and

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} \leq e^{-L\delta^2/2}$$

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Proof of Chernoff Bounds

Markovs Inequality:
Let X be random variable taking non-negative values.
Then
$$\Pr[X \geq a] \leq \mathrm{E}[X]/a$$

Trivial!

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Proof of Chernoff Bounds

Markovs Inequality:
Let X be random variable taking non-negative values.
Then

$$\Pr[X \geq a] \leq \mathrm{E}[X]/a$$

Trivial!

Lemma 87 (Chernoff Bounds)

Let $X_1, \dots, X_n$ be n independent 0-1 random variables, not necessarily identically distributed. Then for $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$, $L \leq \mu \leq U$, and $\delta > 0$

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That's awfully weak :(

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Lemma 89

For $0 \leq \delta \leq 1$ *we have that*

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta^2/3}$$

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As long as derivative of left side is smaller than derivative of right side the inequality holds.

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A convex function ($f''(\delta) \ge 0$) on an interval takes maximum at the boundaries.

$$f'(\delta) = -\frac{1}{1+\delta} + 2/3 \qquad f''(\delta) = \frac{1}{(1+\delta)^2}$$

$f(0) = 0$ and $f(1) = -\ln(2) + 2/3 < 0$

Show:

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$$

Take logarithms:

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True for $\delta = 0$. Divide by U and take derivatives:

$$-\ln(1+\delta) \le -2\delta/3$$

Reason:
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For $\delta \geq 1$ we show

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- Given s_i-t_i pairs in a graph.
- Connect each pair by a path such that not too many path use any given edge.

$$\begin{array}{llrcl} \min & & & W & \\ \text{s.t.} & \forall i & \sum_{p\in\mathcal{P}_i} x_p & = & 1 \\ & & \sum_{p:e\in p} x_p & \le & W \\ & & x_p & \in & \{0,1\} \end{array}$$

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Randomized Rounding:
For each i choose one path from the set $\mathcal{P}_i$ at random according to the probability distribution given by the Linear Programming solution.

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Theorem 90
If $W^* \geq c \ln n$ for some constant c, then with probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + \sqrt{cW^* \ln n}$.

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Let X_e^i be a random variable that indicates whether the path for s_i-t_i uses edge e.

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17.3 MAXSAT

Problem definition:

- n Boolean variables
- m clauses $C_1, \ldots, C_m$. For example

$$C_7 = x_3 \vee \bar{x}_5 \vee \bar{x}_9$$

- Non-negative weight w_j for each clause C_j.
- Find an assignment of true/false to the variables sucht that the total weight of clauses that are satisfied is maximum.

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- Hence, each clause consists of a set of literals (i.e., no duplications: $x_i \vee x_i \vee \bar{x}_j$ is **not** a clause).
- We assume a clause does not contain x_i and $\bar{x}_i$ for any i.
- x_i is called a positive literal while the negation $\bar{x}_i$ is called a negative literal.
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Set each x_i independently to true with probability $\frac{1}{2}$ (and, hence, to false with probability $\frac{1}{2}$, as well).

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Define random variable X_j with

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

Then the total weight W of satisfied clauses is given by

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Then the total weight W of satisfied clauses is given by

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W = \sum_j w_j X_j
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\begin{aligned}
E[W] &= \sum_j w_j E[X_j] \\
&= \sum_j w_j \Pr[C_j \text{ is satisified}] \\
&= \sum_j w_j \Big(1 - \Big(\frac{1}{2}\Big)^{\ell_j}\Big) \\
&\geq \frac{1}{2} \sum_j w_j
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MAXSAT: LP formulation

- Let for a clause C_j, P_j be the set of positive literals and N_j the set of negative literals.

$$C_j = \bigvee_{j \in P_j} x_i \vee \bigvee_{j \in N_j} \bar{x}_i$$

$$\begin{array}{llrcl} \max & & \multicolumn{3}{c}{\sum_j w_j z_j} \\ \text{s.t.} & \forall j & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) & \geq & z_j \\ & \forall i & y_i & \in & \{0, 1\} \\ & \forall j & z_j & \leq & 1 \end{array}$$

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Set each x_i independently to true with probability y_i (and, hence, to false with probability $(1 - y_i)$).

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Lemma 92 (Geometric Mean $\leq$ Arithmetic Mean)

For any nonnegative $a_1, \ldots, a_k$

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} a_i$$

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Definition 93
A function f on an interval I is concave if for any two points s and r from I and any $\lambda \in [0,1]$ we have

$$f(\lambda s + (1-\lambda)r) \geq \lambda f(s) + (1-\lambda) f(r)$$

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$$\begin{aligned} f(\lambda) &= f((1-\lambda)0 + \lambda 1) \\ &\geq (1-\lambda)f(0) + \lambda f(1) \\ &= a + \lambda b \end{aligned}$$

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$\Pr[C_j \text{ not satisfied}]$

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\begin{aligned}
\Pr[C_j \text{ not satisfied}] &= \prod_{i\in P_j}(1-y_i)\prod_{i\in N_j} y_i \\
&\le \left[\frac{1}{\ell_j}\left(\sum_{i\in P_j}(1-y_i)+\sum_{i\in N_j} y_i\right)\right]^{\ell_j} \\
&= \left[1-\frac{1}{\ell_j}\left(\sum_{i\in P_j} y_i+\sum_{i\in N_j}(1-y_i)\right)\right]^{\ell_j}
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The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

$$\Pr[C_j \text{ satisfied}]$$

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\begin{aligned}
\Pr[C_j \text{ not satisfied}] &= \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i \\
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&= \left[1 - \frac{1}{\ell_j} \left(\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \right) \right]^{\ell_j} \\
&\le \left(1 - \frac{z_j}{\ell_j} \right)^{\ell_j} .
\end{aligned}
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The function $f(z) = 1 - (1 - \frac{z}{\ell})^\ell$ is concave. Hence,

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\Pr[C_j \text{ not satisfied}] &= \prod_{i\in P_j}(1-y_i)\prod_{i\in N_j} y_i \\
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The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

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\Pr[C_j \text{ satisfied}] &\geq 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j} \\
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$f''(z) = -\frac{\ell-1}{\ell}\left[1 - \frac{z}{\ell}\right]^{\ell-2} \leq 0$ for $z \in [0,1]$. Therefore, f is concave.

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\end{aligned}
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$$E[W]$$

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$$E[W]=\sum_j w_j \Pr[C_j \text{ is satisfied}]$$

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MAXSAT: The better of two

Theorem 95
Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$-approximation.

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\begin{aligned}
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&\geq \sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \\
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Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$

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Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$
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E[\max\{W_1, W_2\}] & \\
&\geq E[\tfrac{1}{2}W_1 + \tfrac{1}{2}W_2] \\
&\geq \frac{1}{2}\sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_j w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)
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&\qquad \geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1-\left(1-\frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq\frac{3}{4}\text{ for all integers}}\Bigg]
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&\quad\geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq \frac{3}{4}\text{for all integers}}\Bigg]\\
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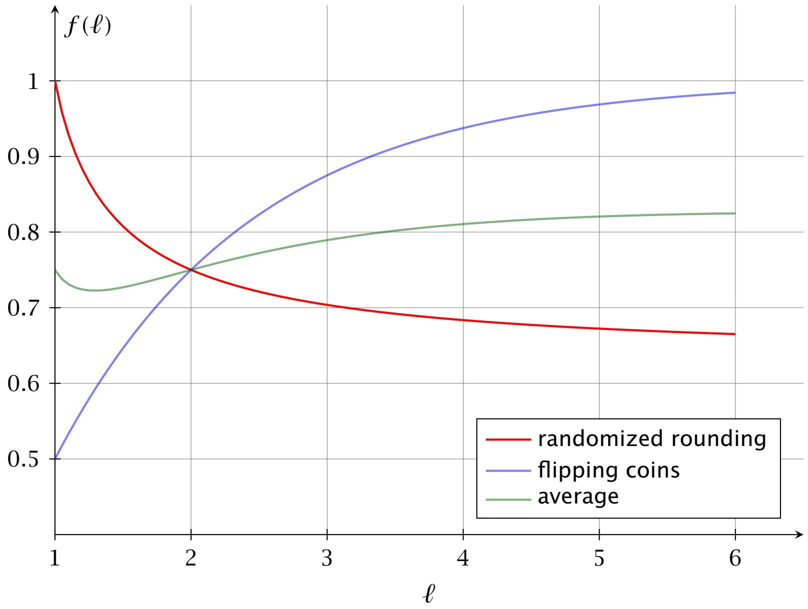

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&\quad \geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq \frac{3}{4} \text{for all integers}}\Bigg] \\
&\quad \geq \frac{3}{4}\text{OPT}
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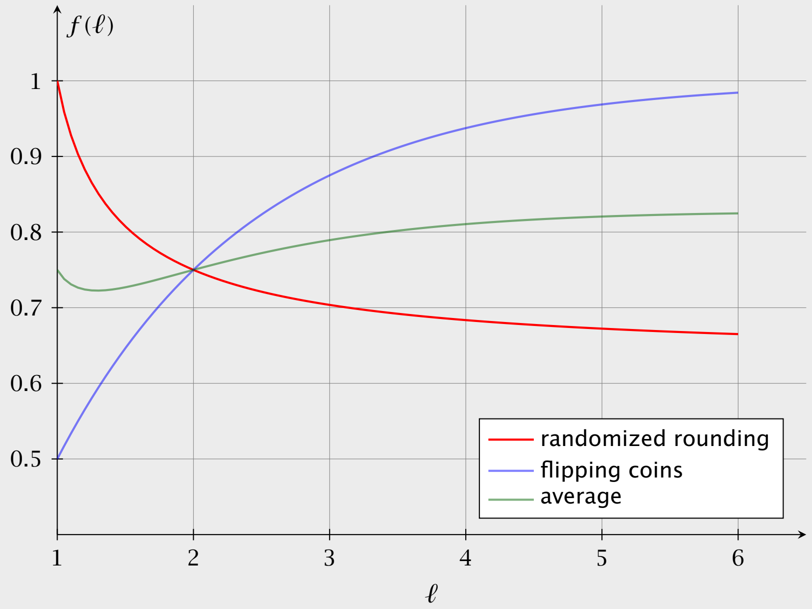
MAXSAT: Nonlinear Randomized Rounding

So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

We could define a function $f : [0,1] \to [0,1]$ and set x_i to true with probability $f(y_i)$.



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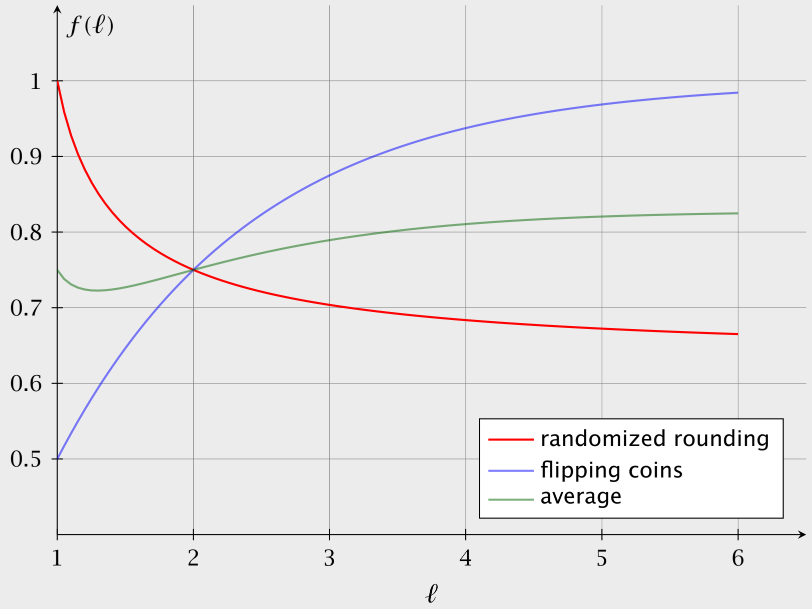

MAXSAT: Nonlinear Randomized Rounding

Let $f : [0,1] \to [0,1]$ be a function with

$$1 - 4^{-x} \le f(x) \le 4^{x-1}$$

Theorem 96

Rounding the LP-solution with a function f of the above form gives a $\frac{3}{4}$-approximation.

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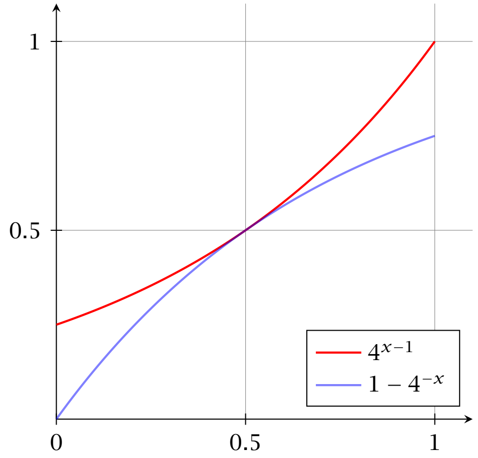
$$1 - 4^{-x} \le f(x) \le 4^{x-1}$$

Theorem 96
Rounding the LP-solution with a function f of the above form gives a $\frac{3}{4}$-approximation.

MAXSAT: Nonlinear Randomized Rounding

So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

We could define a function $f : [0,1] \to [0,1]$ and set x_i to true with probability $f(y_i)$.



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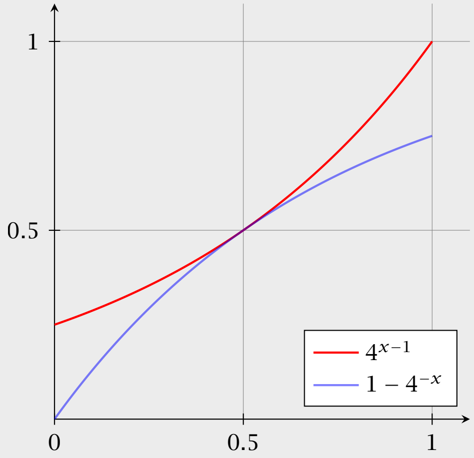
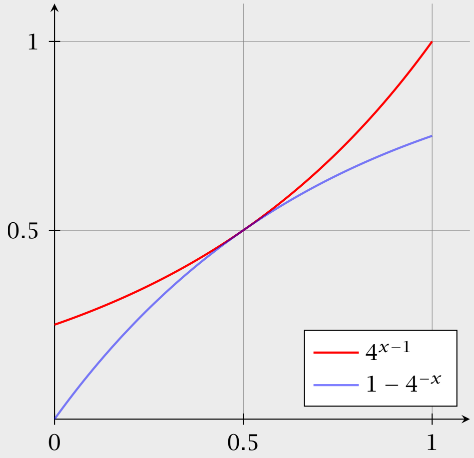
$\Pr[C_j \text{ not satisfied}]$



$$\Pr[C_j \text{ not satisfied}] = \prod_{i\in P_j}(1-f(y_i)) \prod_{i\in N_j} f(y_i)$$



$$\Pr[C_j \text{ not satisfied}] = \prod_{i\in P_j}(1-f(y_i))\prod_{i\in N_j} f(y_i)$$
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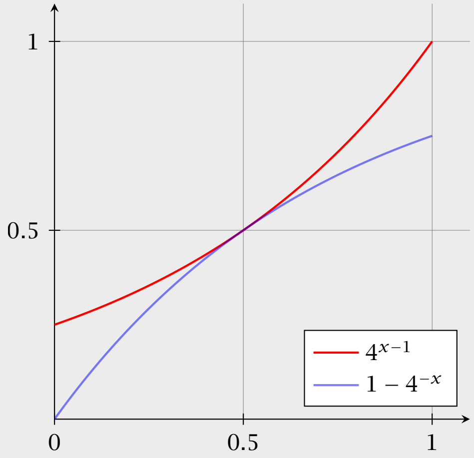

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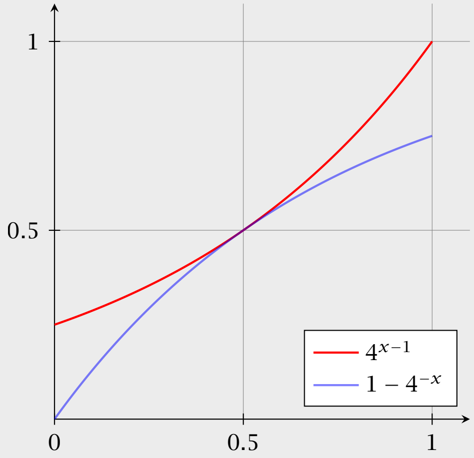

EADS II
Harald Räcke
17.3 MAXSAT

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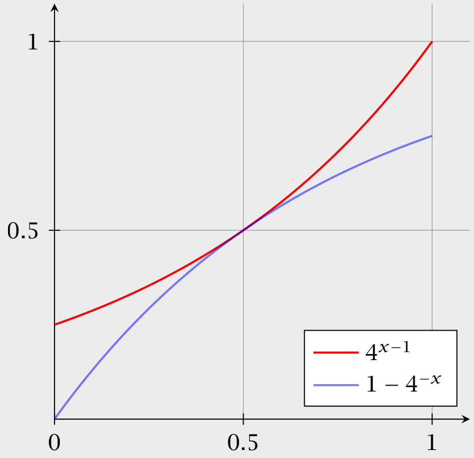

The function $g(z) = 1 - 4^{-z}$ is concave on $[0,1]$. Hence,

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Can we do better?

Not if we compare ourselves to the value of an optimum LP-solution.

Definition 97 (Integrality Gap)

The integrality gap for an ILP is the worst-case ratio over all instances of the problem of the value of an optimal IP-solution to the value of an optimal solution to its linear programming relaxation.

Note that the integrality is less than one for maximization problems and larger than one for minimization problems (of course, equality is possible).

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Our ILP-formulation for the MAXSAT problem has integrality gap at most $\frac{3}{4}$.

$$\begin{array}{lllll} \max & & \sum_j w_j z_j & & \\ \text{s.t.} & \forall j & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) & \geq & z_j \\ & \forall i & y_i & \in & \{0,1\} \\ & \forall j & z_j & \leq & 1 \end{array}$$

Consider: $(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$

- any solution can satisfy at most 3 clauses
- we can set $y_1 = y_2 = 1/2$ in the LP; this allows to set $z_1 = z_2 = z_3 = z_4 = 1$
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MaxCut

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Given a weighted graph $G = (V, E, w)$, $w(v) \geq 0$, partition the vertices into two parts. Maximize the weight of edges between the parts.

Trivial 2-approximation

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Semidefinite Programming

$$\begin{array}{lrrcl}
\max / \min & & \sum_{i,j} c_{ij} x_{ij} & & \\
\text{s.t.} & \forall k & \sum_{i,j,k} a_{ijk} x_{ij} & = & b_k \\
 & \forall i,j & x_{ij} & = & x_{ji} \\
 & & X = (x_{ij}) \text{ is psd.} & &
\end{array}$$

- linear objective, linear contraints
- we can constrain a square matrix of variables to be symmetric positive definite

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- variables are vectors in n-dimensional space
- objective functions and contraints are linear in inner products of the vectors

This is equivalent!

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Fact [without proof]
We (essentially) can solve Semidefinite Programs in polynomial time...

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Quadratic Programs

Quadratic Program for MaxCut:

$$\begin{array}{lrcl} \max & \frac{1}{2}\sum_{i,j} w_{ij}(1 - y_i y_j) & & \\ \forall i & y_i & \in & \{-1, 1\} \end{array}$$

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Semidefinite Relaxation

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- this is clearly a relaxation
- the solution will be vectors on the unit sphere

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Rounding the SDP-Solution

- Choose a random vector r such that $r/\|r\|$ is uniformly distributed on the unit sphere.
- If $r^t v_i > 0$ set $y_i = 1$ else set $y_i = -1$

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Rounding the SDP-Solution

Choose the i-th coordinate r_i as a Gaussian with mean 0 and variance 1, i.e., $r_i \sim \mathcal{N}(0,1)$.

Density function:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{x^2/2}$$

Then

$$\begin{aligned}\Pr[r = (x_1, \dots, x_n)] &\\ &= \frac{1}{(\sqrt{2\pi})^n} e^{x_1^2/2} \cdot e^{x_2^2/2} \cdot \ldots \cdot e^{x_n^2/2}\, dx_1 \cdot \ldots \cdot dx_n \\ &= \frac{1}{(\sqrt{2\pi})^n} e^{\frac{1}{2}(x_1^2+\ldots+x_n^2)}\, dx_1 \cdot \ldots \cdot dx_n\end{aligned}$$

Hence the probability for a point only depends on its distance to the origin.

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Fact

The projection of r onto two unit vectors e_1 and e_2 are independent and are normally distributed with mean 0 and variance 1 iff e_1 and e_2 are orthogonal.

Note that this is clear if e_1 and e_2 are standard basis vectors.

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Corollary
If we project r onto a hyperplane its normalized projection ($r'/\|r'\|$) is uniformly distributed on the unit circle within the hyperplane.

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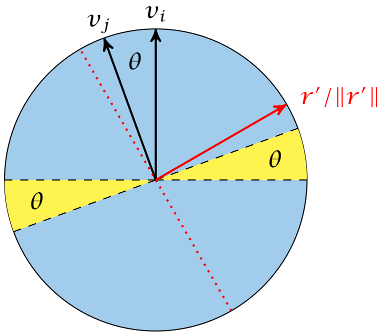

- if the normalized projection falls into the shaded region, v_i and v_j are rounded to different values
- this happens with probability θ/π

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Corollary
If we project r onto a hyperplane its normalized projection ($r'/\|r'\|$) is uniformly distributed on the unit circle within the hyperplane.

Rounding the SDP-Solution

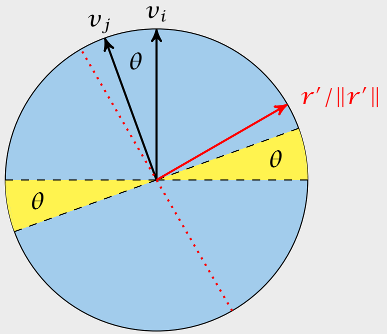
- contribution of edge (i,j) to the SDP-relaxation:

$$\frac{1}{2}w_{ij}\left(1-v_i^t v_j\right)$$

- (expected) contribution of edge (i,j) to the rounded instance $w_{ij}\arccos(v_i^t v_j)/\pi$
- ratio is at most

$$\min_{x\in[-1,1]}\frac{2\arccos(x)}{\pi(1-x)}\geq 0.878$$

Rounding the SDP-Solution



- if the normalized projection falls into the shaded region, v_i and v_j are rounded to different values
- this happens with probability θ/π

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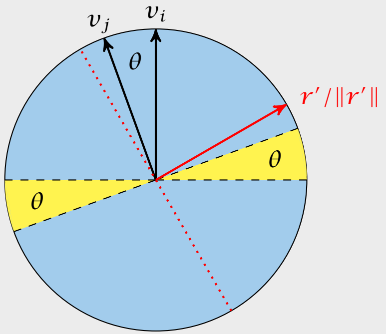

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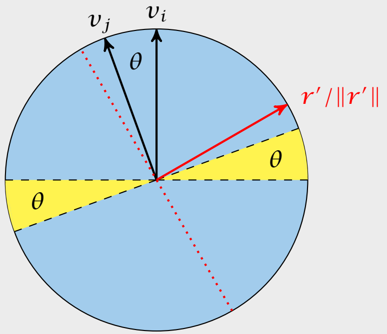

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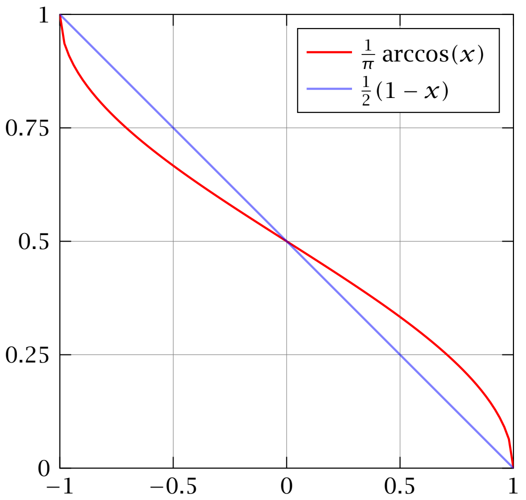

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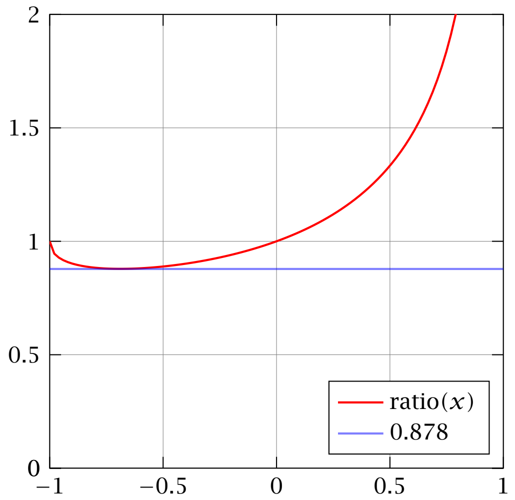

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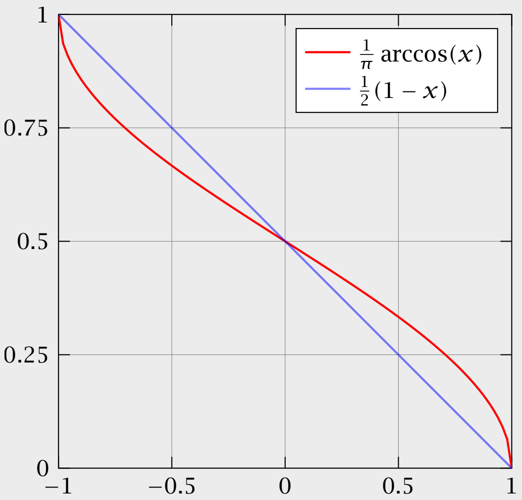


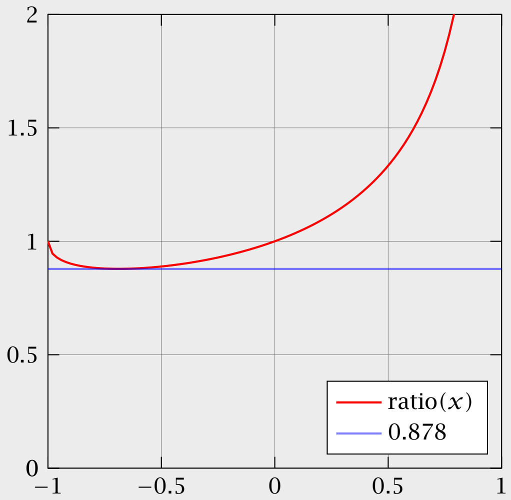
Rounding the SDP-Solution

Theorem 99
Given the unique games conjecture, there is no α-approximation for the maximum cut problem with constant

$$\alpha > \min_{x\in[-1,1]} \frac{2\arccos(x)}{\pi(1-x)}$$

unless $\mathrm{P} = \mathrm{NP}$.

Rounding the SDP-Solution



Repetition: Primal Dual for Set Cover

Primal Relaxation:

$$\begin{array}{llrcl} \min & & \sum_{i=1}^{k} w_i x_i & & \\ \text{s.t.} & \forall u \in U & \sum_{i:u\in S_i} x_i & \geq & 1 \\ & \forall i \in \{1,\dots,k\} & x_i & \geq & 0 \end{array}$$

Dual Formulation:

$$\begin{array}{llrcl} \max & & \sum_{u\in U} y_u & & \\ \text{s.t.} & \forall i \in \{1,\dots,k\} & \sum_{u:u\in S_i} y_u & \leq & w_i \\ & & y_u & \geq & 0 \end{array}$$

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Note that the constructed pair of primal and dual solution fulfills primal slackness conditions.

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This means

$$x_j > 0 \Rightarrow \sum_{e \in S_j} y_e = w_j$$

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$$y_e > 0 \Rightarrow \sum_{j : e \in S_j} x_j = 1$$

then the solution would be optimal!!!

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We don’t fulfill these constraint but we fulfill an approximate version:

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primal cost

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$$\underbrace{\sum_j \boxed{c_j}\, x_j}_{\text{primal cost}}$$

right hand side of j-th dual constraint

primal cost

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&= \alpha \sum_i \left(\sum_j a_{ij} x_j \right) y_i \\
&\leq \alpha\beta \cdot \sum_i b_i y_i
\end{aligned}
$$

Suppose we have a primal/dual pair

$$
\begin{array}{llrcl}
\min & & \sum_j c_j x_j & & \\
\text{s.t.} & \forall i & \sum_{j:} a_{ij} x_j & \geq & b_i \\
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\end{array}
\qquad
\begin{array}{llrcl}
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and solutions that fulfill approximate slackness conditions:

$$x_j > 0 \Rightarrow \sum_i a_{ij} y_i \geq \frac{1}{\alpha} c_j$$

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Then

$$\boxed{\sum_j c_j x_j} \le \alpha \sum_j \left(\sum_i a_{ij} y_i \right) x_j$$

primal cost

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dual objective

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Feedback Vertex Set for Undirected Graphs

- Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.

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Primal Relaxation:

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In any graph with no vertices of degree 1, there always exists a cycle that has at most $\mathcal{O}(\log n)$ vertices of degree 3 or more. We can find such a cycle in linear time.

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Primal Dual for Shortest Path

Given a graph $G = (V, E)$ with two nodes $s, t \in V$ and edge-weights $c : E \to \mathbb{R}^+$ find a shortest path between s and t w.r.t. edge-weights c.

$$\begin{array}{llrcl} \min & & \sum_e c(e)x_e & & \\ \text{s.t.} & \forall S \in \mathcal{S} & \sum_{e:\delta(S)} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

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1: $y \leftarrow 0$
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y ← 0
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If we can show that $y_S > 0$ implies $|P\cap\delta(S)| = 1$ gives

$$\sum_{e\in P} c(e) = \sum_S y_S \le \mathrm{OPT}$$

by weak duality.

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- In each iteration we take the current connected component from (V,F) that contains s (call this component C) and add some edge from $\delta(C)$ to F.
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$$\sum_{e\in P} c(e) = \sum_S y_S \le \mathrm{OPT}$$

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Given a graph $G = (V, E)$, together with source-target pairs s_i, t_i, $i = 1, \dots, k$, and a cost function $c : E \to \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that for every $i \in \{1, \dots, k\}$ there is a path between s_i and t_i only using edges in F.

$$\begin{array}{llrcl} \min & & \sum_e c(e)x_e & & \\ \text{s.t.} & \forall S \subseteq V : S \in S_i \text{ for some } i & \sum_{e \in \delta(S)} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

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The difference to the dual of the shortest path problem is that we have many more variables (sets for which we can generate a moat of non-zero width).

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```
y ← 0; F ← ∅; ℓ ← 0
while not all s_i-t_i pairs connected in F do
    ℓ ← ℓ + 1
    Let ℭ be set of all connected components C of (V,F)
    such that |C ∩ {s_i, t_i}| = 1 for some i.
    Increase y_C for all C ∈ ℭ uniformly until for some edge
    e_ℓ ∈ δ(C'), C' ∈ ℭ s.t. ∑_{S:e_ℓ∈δ(S)} y_S = c_{e_ℓ}
    F ← F ∪ {e_ℓ}
F' ← F
for k ← ℓ downto 1 do // reverse deletion
    if F' − e_k is feasible solution then
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The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.

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$y \leftarrow 0$; $F \leftarrow \emptyset$; $\ell \leftarrow 0$
while not all s_i-t_i pairs connected in F **do**
  $\ell \leftarrow \ell + 1$
  Let $\mathbb{C}$ be set of all connected components C of (V, F) such that $|C \cap \{s_i, t_i\}| = 1$ for some i.
  Increase y_C for all $C \in \mathbb{C}$ uniformly until for some edge $e_\ell \in \delta(C')$, $C' \in \mathbb{C}$ s.t. $\sum_{S : e_\ell \in \delta(S)} y_S = c_{e_\ell}$
  $F \leftarrow F \cup \{e_\ell\}$
$F' \leftarrow F$
for $k \leftarrow \ell$ downto 1 **do** // reverse deletion
  if $F' - e_k$ is feasible solution **then**
    remove e_k from F'
return F'

Example

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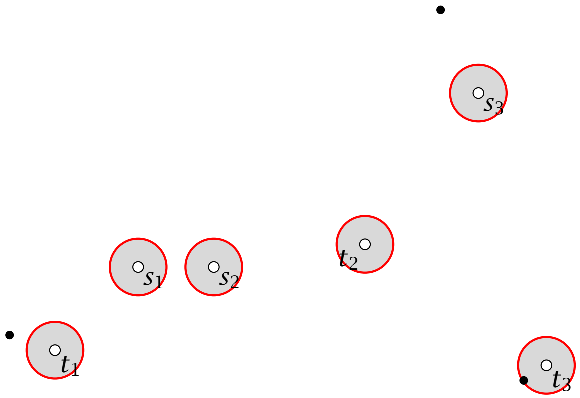
Example



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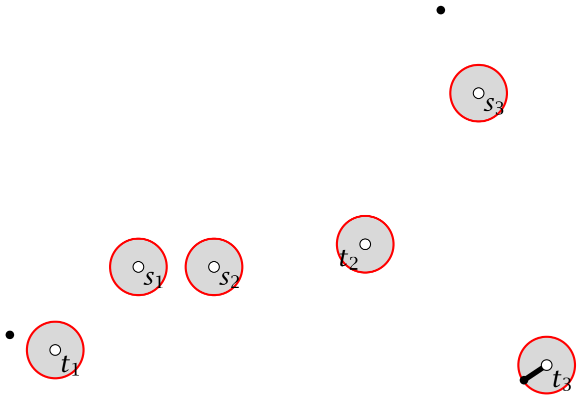

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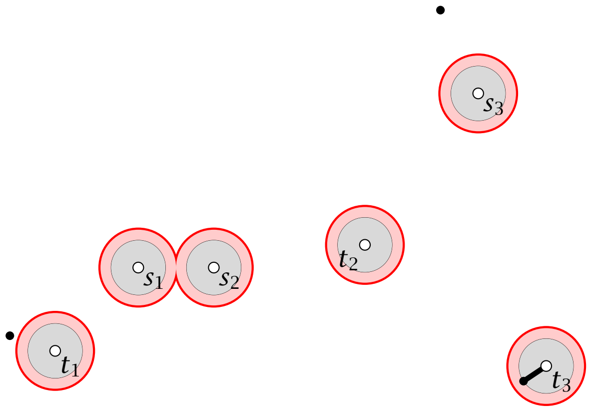

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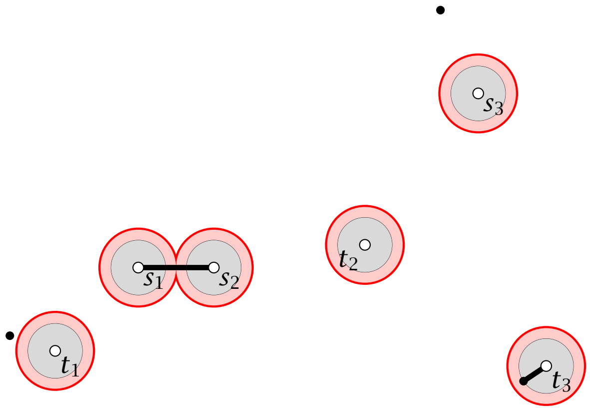

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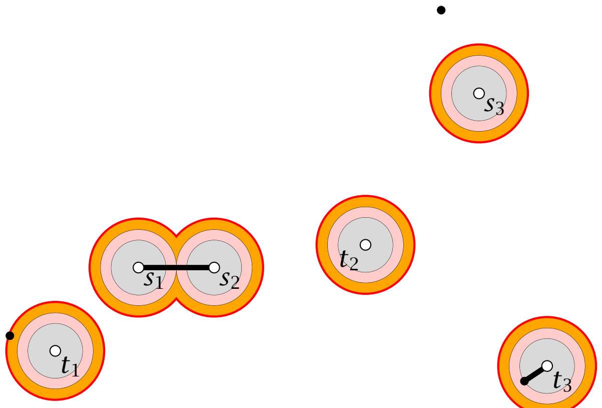

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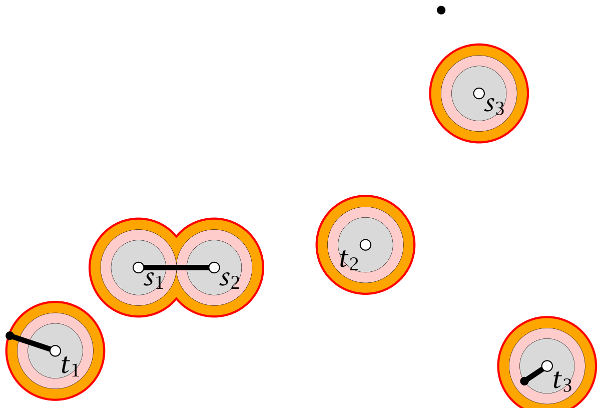

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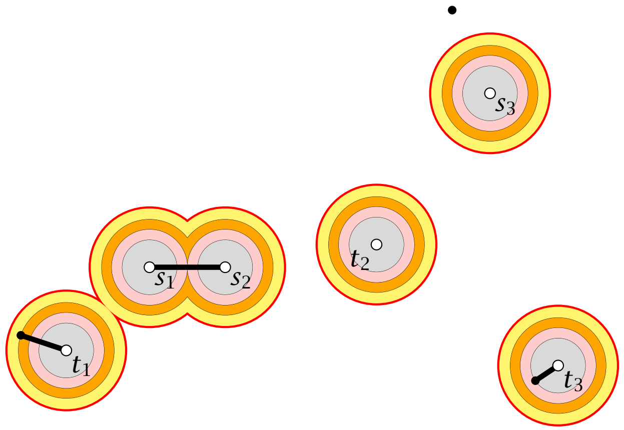

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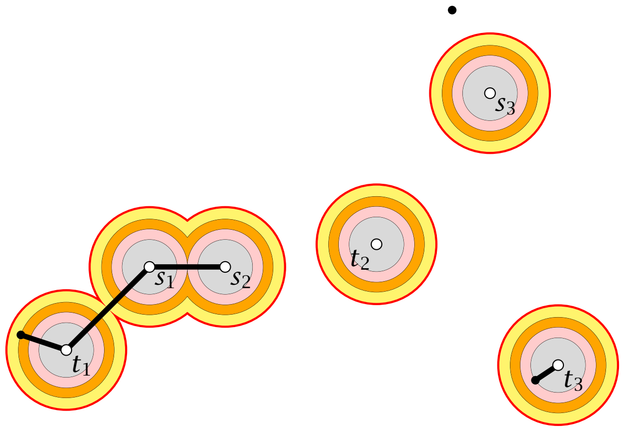

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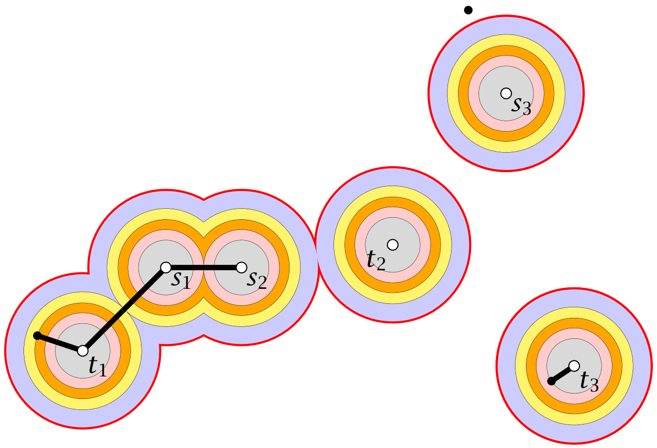

The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.

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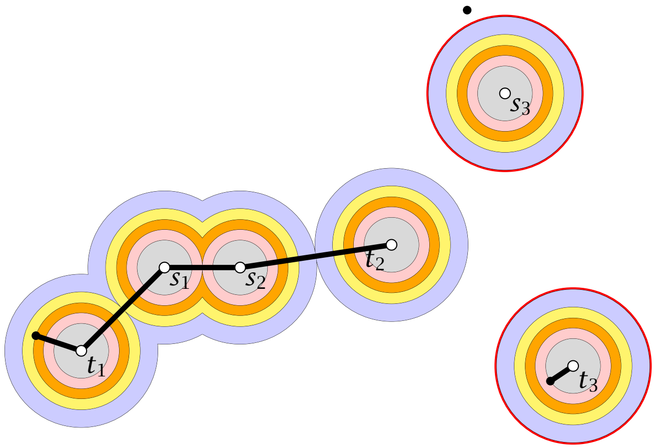

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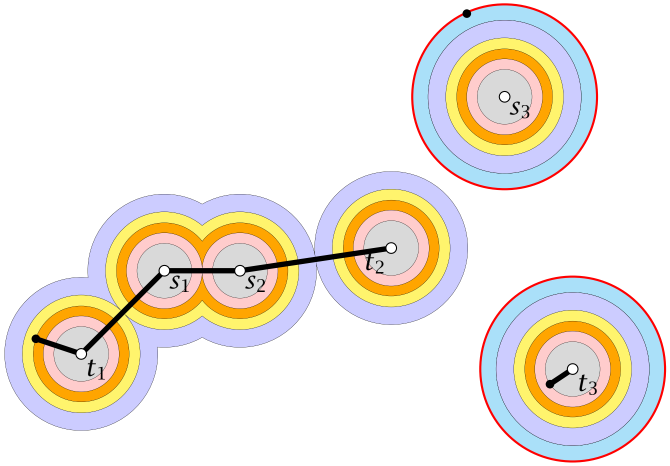

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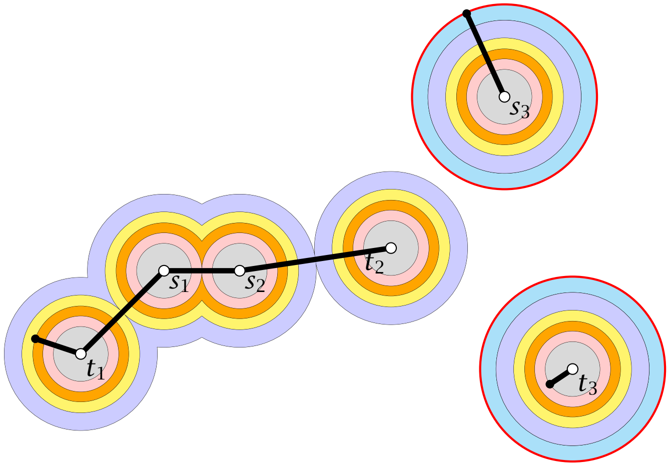

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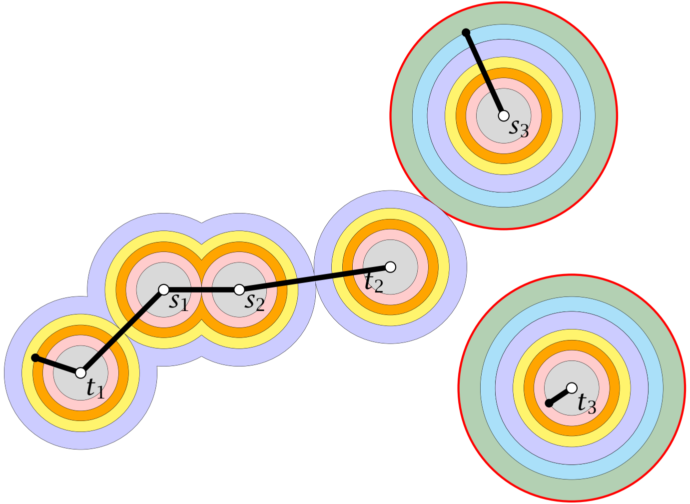

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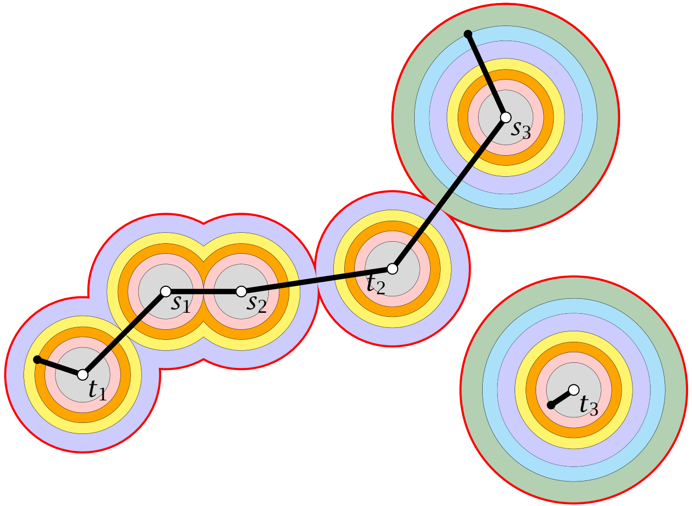

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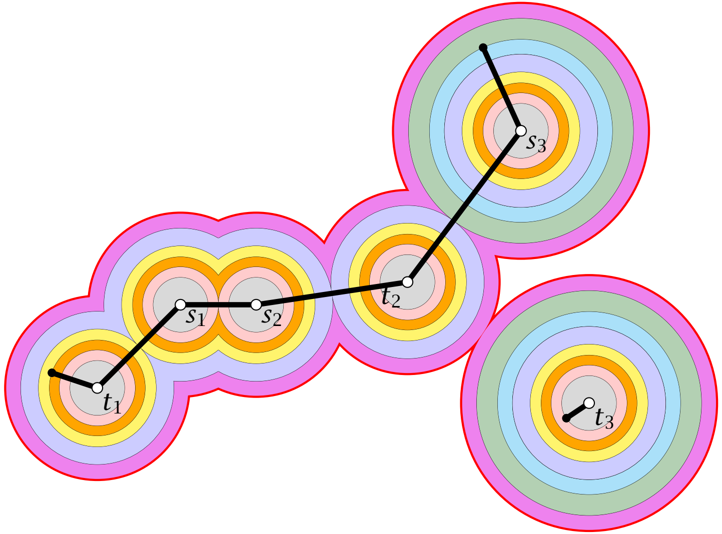

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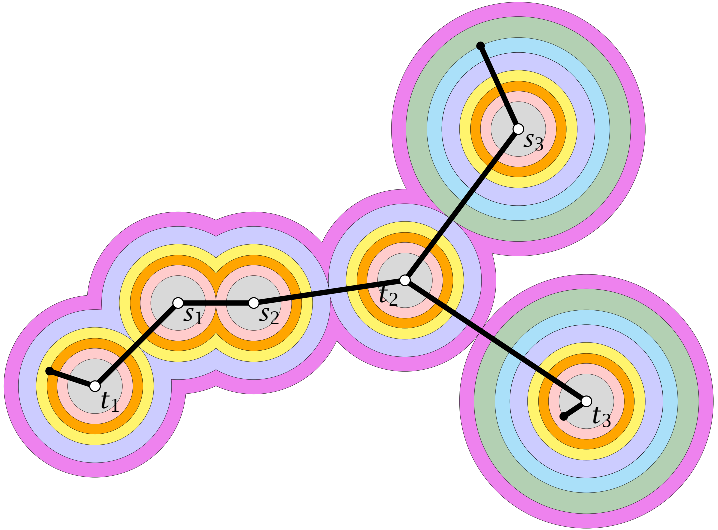

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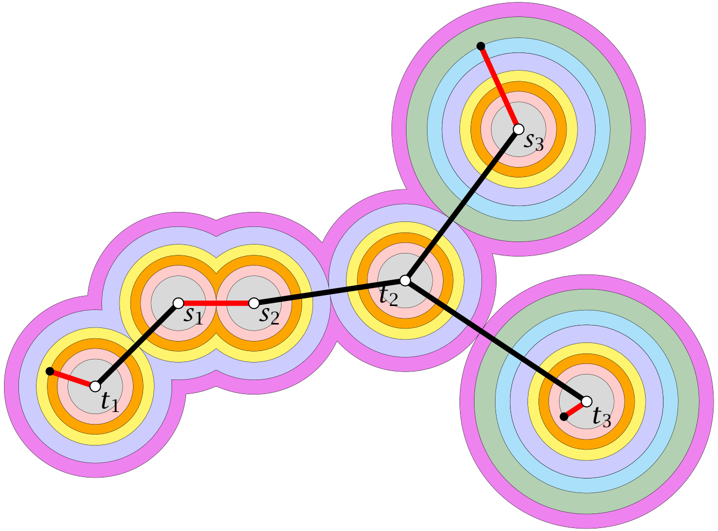

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Lemma 102
For any $\mathfrak{C}$ in any iteration of the algorithm

$$\sum_{C\in\mathfrak{C}} |\delta(C)\cap F'| \le 2|\mathfrak{C}|$$

This means that the number of times a moat from $\mathfrak{C}$ is crossed in the final solution is at most twice the number of moats.

Proof: later...

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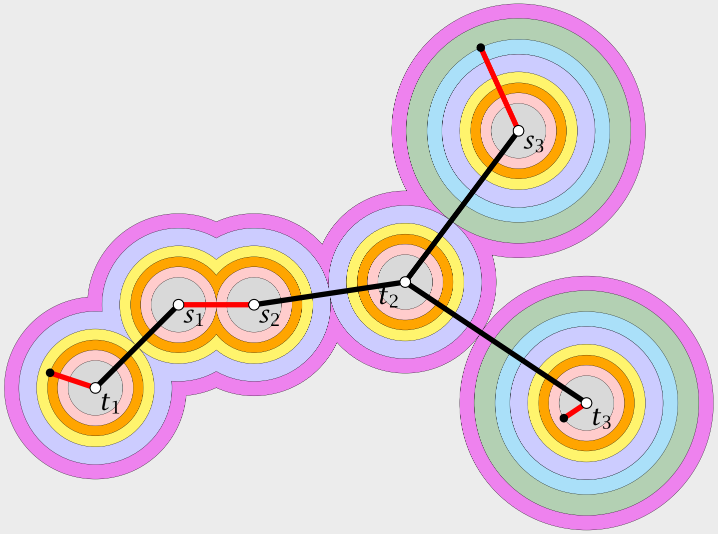

$$\sum_{e\in F'} c_e = \sum_{e\in F'} \sum_{S: e\in\delta(S)} y_S = \sum_S |F' \cap \delta(S)| \cdot y_S \enspace .$$

We want to show that

$$\sum_S |F' \cap \delta(S)| \cdot y_S \le 2 \sum_S y_S$$

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  and the increase of the right hand side is $2\epsilon|\mathfrak{C}|$.
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- Contract all edges in F_i into single vertices V'.
- We can consider the forest H on the set of vertices V'.
- Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- Color a vertex $v \in V'$ red if it corresponds to a component from $\mathfrak{C}$ (an active component). Otw. color it blue. (Let B the set of blue vertices (with non-zero degree) and R the set of red vertices)
- We have

$$\sum_{v\in R} \deg(v) \ge \sum_{C\in\mathfrak{C}} |\delta(C)\cap F'| \overset{?}{\le} 2|\mathfrak{C}| = 2|R|$$

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- Suppose that no node in B has degree one.

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$$\begin{aligned}\sum_{v\in R}\deg(v) &= \sum_{v\in R\cup B}\deg(v) - \sum_{v\in B}\deg(v)\\ &\le 2(|R|+|B|) - 2|B| = 2|R|\end{aligned}$$

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- Every blue vertex with non-zero degree must have degree at least two.

- Contract all edges in F_i into single vertices V'.
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  - Suppose not. The single edge connecting $b \in B$ comes from H, and, hence, is necessary.
  - But this means that the cluster corresponding to b must separate a source-target pair.
  - But then it must be a red node.

- Contract all edges in F_i into single vertices V'.
- We can consider the forest H on the set of vertices V'.
- Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- Color a vertex $v \in V'$ red if it corresponds to a component from $\mathfrak{C}$ (an active component). Otw. color it blue. (Let B the set of blue vertices (with non-zero degree) and R the set of red vertices)
- We have

$$\sum_{v\in R}\deg(v) \ge \sum_{C\in\mathfrak{C}}|\delta(C)\cap F'| \overset{?}{\le} 2|\mathfrak{C}| = 2|R|$$

19 Cuts & Metrics

Shortest Path

$$\begin{array}{llrcl} \min & & \sum_e c(e) x_e & & \\ \text{s.t.} & \forall S \in \mathcal{S} & \sum_{e \in \delta(S)} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

$\mathcal{S}$ is the set of subsets that separate s from t.

The Dual:

$$\begin{array}{llrcl} \max & & \sum_S y_S & & \\ \text{s.t.} & \forall e \in E & \sum_{S: e \in \delta(S)} y_S & \leq & c(e) \\ & \forall S \in \mathcal{S} & y_S & \geq & 0 \end{array}$$

The Separation Problem for the Shortest Path LP is the Minimum Cut Problem.

- Suppose that no node in B has degree one.
- Then

$$\sum_{v \in R} \deg(v) = \sum_{v \in R \cup B} \deg(v) - \sum_{v \in B} \deg(v)$$
$$\leq 2(|R| + |B|) - 2|B| = 2|R|$$

- Every blue vertex with non-zero degree must have degree at least two.
  - Suppose not. The single edge connecting $b \in B$ comes from H, and, hence, is necessary.
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19 Cuts & Metrics

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$\mathcal{P}$ is the set of path that connect s and t.

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19 Cuts & Metrics

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19 Cuts & Metrics

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How do we round the LP?

- Let $B(s,r)$ be the ball of radius r around s (w.r.t. metric d). Formally:

$$B = \{v \in V \mid d(s,v) \le r\}$$

- For $0 \le r < 1$, $B(s,r)$ is an s-t-cut.

Which value of r should we choose? choose randomly!!!

Formally:
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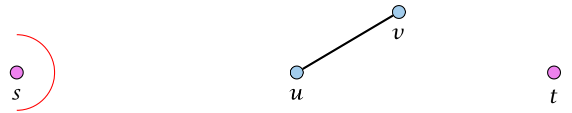
What is the probability that an edge (u, v) is in the cut?



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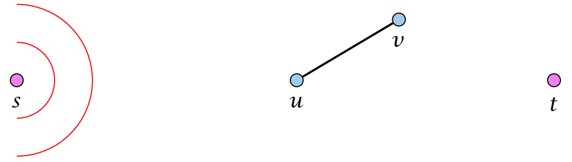

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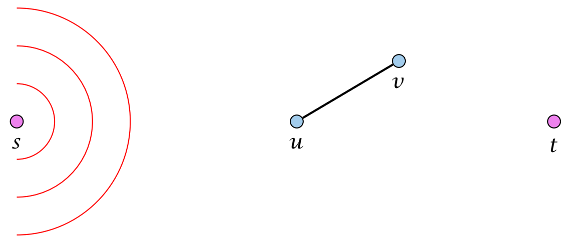

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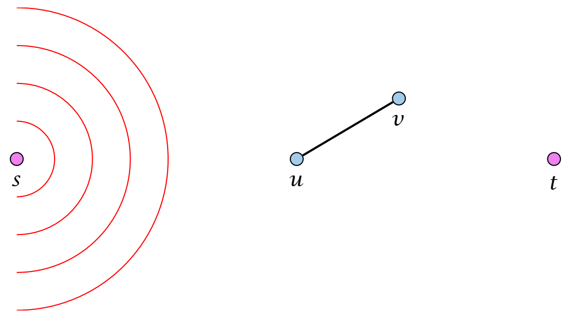

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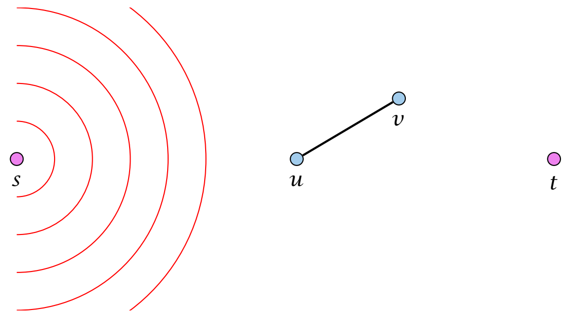

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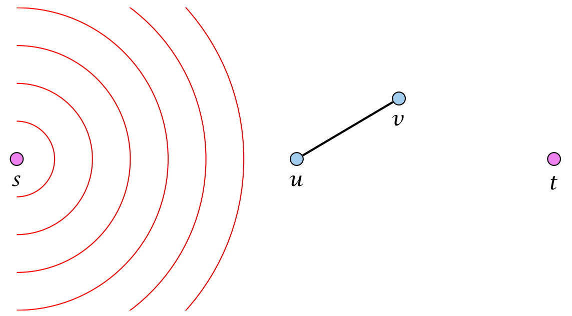

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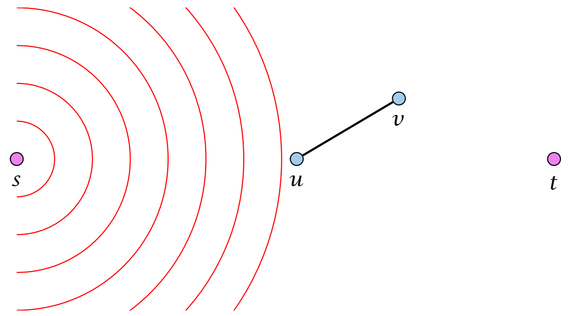

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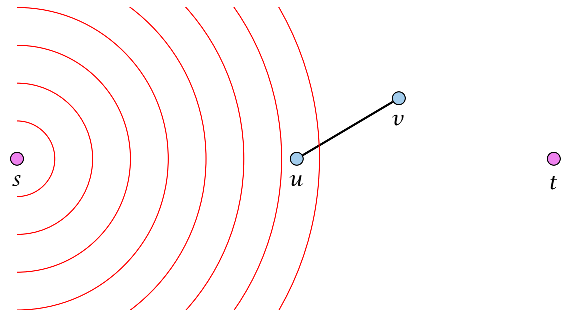

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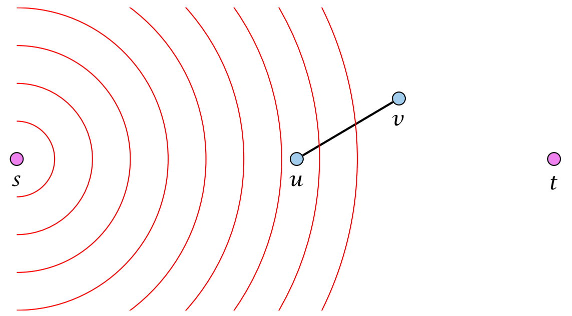

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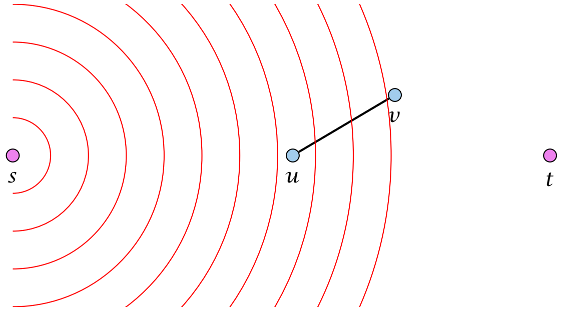

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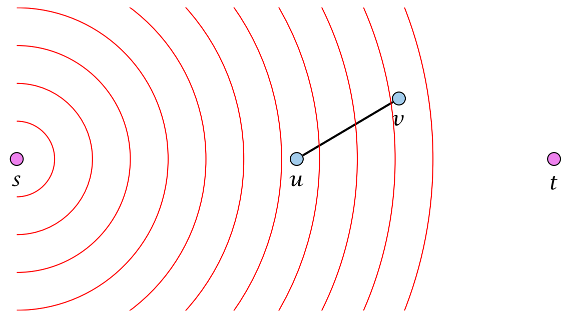

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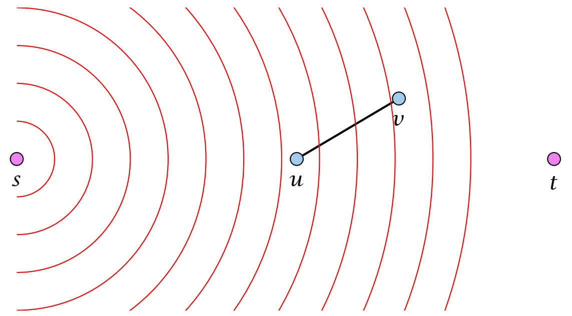

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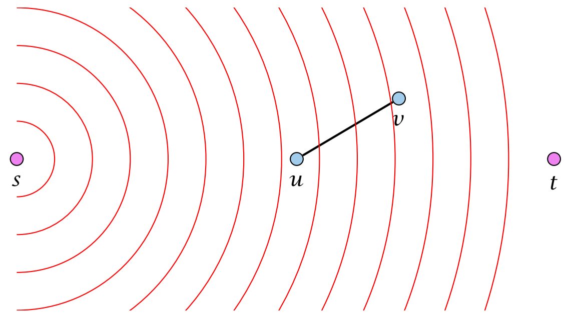

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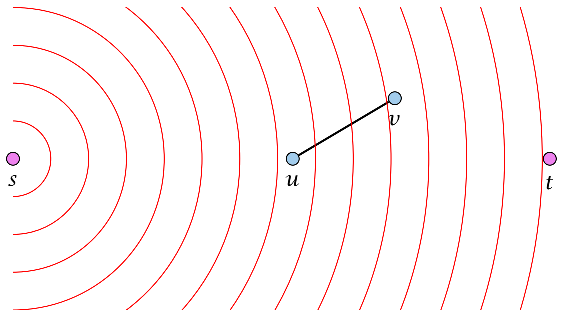

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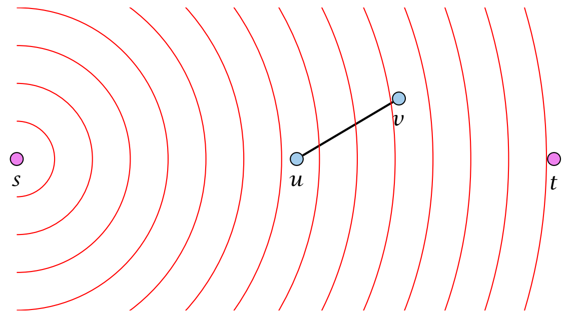

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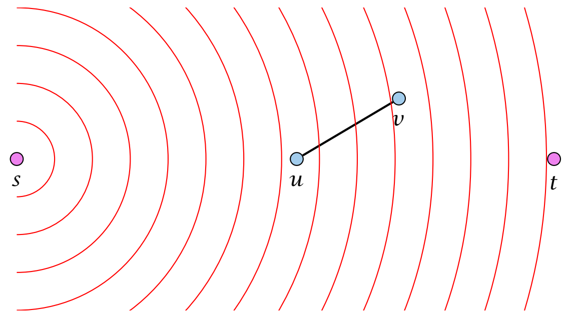

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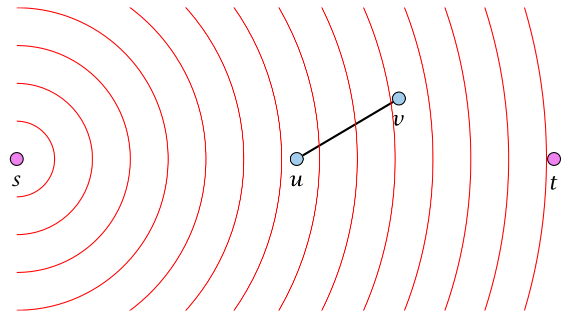

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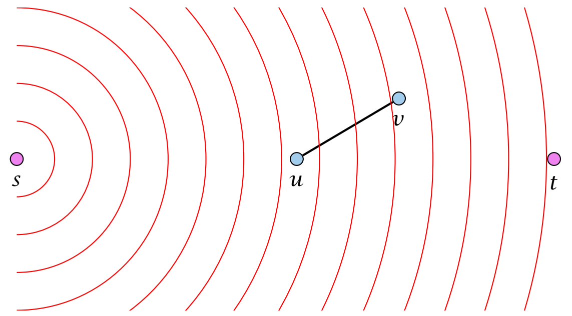

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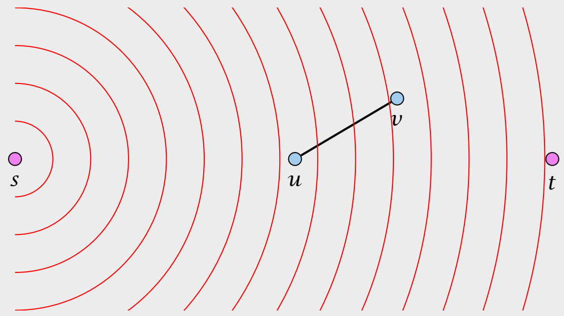

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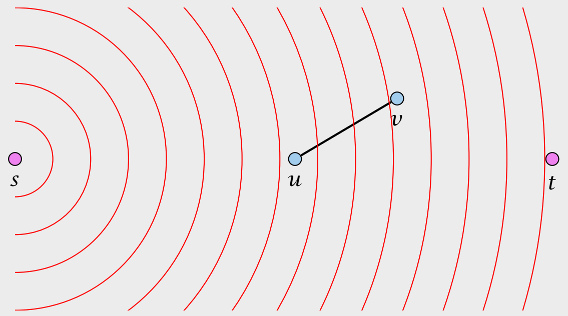

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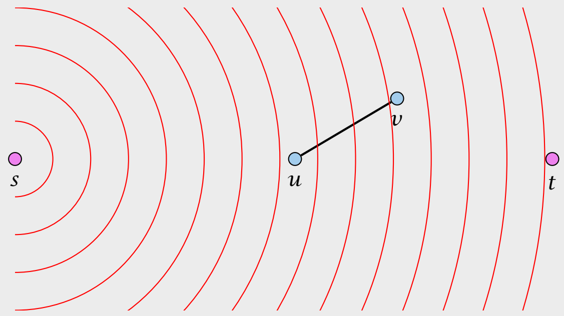

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- Assume for simplicity that all edge-length ℓ_e are multiples of $\delta \ll 1$.
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Algorithm 1 RegionGrowing(s_i, p)

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z ← 0
repeat
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until heads
return B(s_i, z)
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while $\exists s_i$-t_i pair in G' **do**
  $C \leftarrow \text{RegionGrowing}(s_i, p)$
  $G' = G' \setminus C$ // cuts edges leaving C
return $B(s_i, z)$

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  flip a coin ($\Pr[\text{heads}] = p$)
  $z \leftarrow z + 1$
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return $B(s_i, z)$

Algorithm 1 Multicut(G')

while $\exists s_i$-t_i pair in G' **do**
    $C \leftarrow$ RegionGrowing(s_i, p)
    $G' = G' \setminus C$ // cuts edges leaving C
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- probability of cutting an edge is only p
- a source either does not reach an edge during Region Growing; then it is not cut
- if it reaches the edge then it either cuts the edge or protects the edge from being cut by other sources
- if we choose $p = \delta$ the probability of cutting an edge is only its LP-value; our expected cost are at most OPT.

- Assume for simplicity that all edge-length ℓ_e are multiples of $\delta \ll 1$.
- Replace the graph G by a graph G', where an edge of length ℓ_e is replaced by ℓ_e/δ edges of length δ.
- Let $B(s_i, z)$ be the ball in G' that contains nodes v with distance $d(s_i, v) \le z\delta$.

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Problem:

We may not cut all source-target pairs.

A component that we remove may contain an s_i-t_i pair.

If we ensure that we cut before reaching radius $1/2$ we are in good shape.

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- we say a Region Growing is not successful if it does not terminate before reaching radius $1/2$.

$$\Pr[\text{not successful}] \le (1-p)^{\frac{1}{2\delta}} = \left((1-p)^{1/p}\right)^{\frac{p}{2\delta}} \le e^{-\frac{p}{2\delta}} \le \frac{1}{k^3}$$

- Hence,

$$\Pr[\exists i \text{ that is not successful}] \le \frac{1}{k^2}$$

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What is expected cost?

$$\begin{aligned}\text{E}[\text{cutsize}] &= \Pr[\text{success}] \cdot \text{E}[\text{cutsize} \mid \text{success}] \\ &\quad + \Pr[\text{no success}] \cdot \text{E}[\text{cutsize} \mid \text{no success}]\end{aligned}$$

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$$\mathrm{E}[\text{cutsize}] = \Pr[\text{success}] \cdot \mathrm{E}[\text{cutsize} \mid \text{success}] + \Pr[\text{no success}] \cdot \mathrm{E}[\text{cutsize} \mid \text{no success}]$$

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If we are not successful we simply perform a trivial k-approximation.

This only increases the expected cost by at most $\frac{1}{k^2} \cdot k\mathrm{OPT} \le \mathrm{OPT}/k$.

Hence, our final cost is $\mathcal{O}(\ln k) \cdot \mathrm{OPT}$ in expectation.

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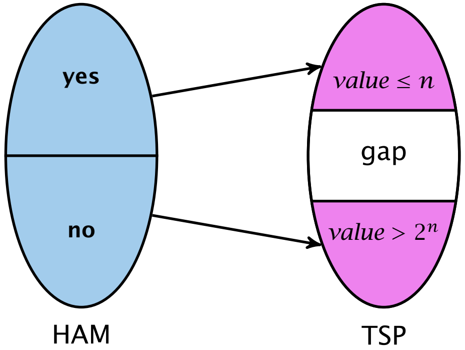
Gap Introducing Reduction



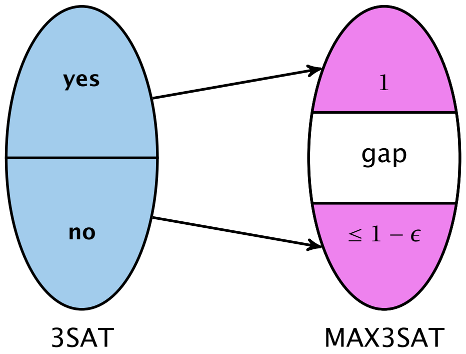
Reduction from Hamiltonian cycle to TSP

- instance that has Hamiltonian cycle is mapped to TSP instance with small cost
- otherwise it is mapped to instance with large cost
- $\Longrightarrow$ there is no $2^n/n$-approximation for TSP

PCP theorem: Approximation View

Theorem 104 (PCP Theorem A)

There exists $\epsilon > 0$ for which there is gap introducing reduction between 3SAT and MAX3SAT.



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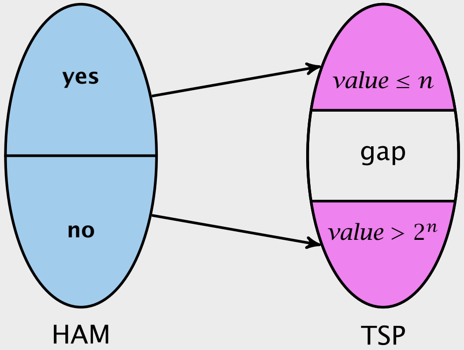

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PCP theorem: Proof System View

Definition 105 (NP)

A language $L \in \mathrm{NP}$ if there exists a polynomial time, deterministic verifier V (a Turing machine), s.t.

[$x \in L$] completeness
There exists a proof string y, $|y| = \mathrm{poly}(|x|)$, s.t. $V(x, y) =$ "accept".

[$x \notin L$] soundness
For any proof string y, $V(x, y) =$ "reject".

Note that requiring $|y| = \mathrm{poly}(|x|)$ for $x \notin L$ does not make a difference (**why?**).

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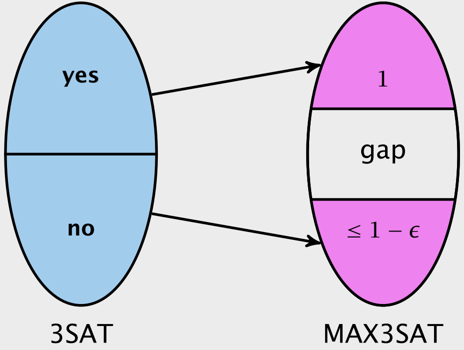

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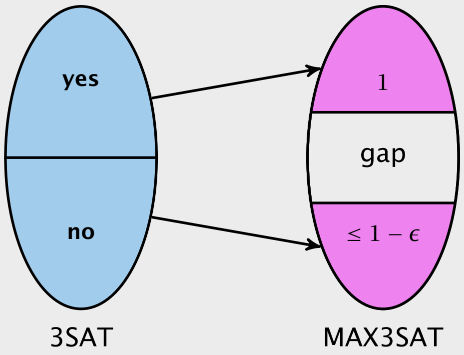

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Probabilistic Checkable Proofs

An Oracle Turing Machine M is a Turing machine that has access to an oracle.

Such an oracle allows M to solve some problem in a single step.

For example having access to a TSP-oracle π_{TSP} would allow M to write a TSP-instance x on a special oracle tape and obtain the answer (yes or no) in a single step.

For such TMs one looks in addition to running time also at query complexity, i.e., how often the machine queries the oracle.

For a proof string y, π_y is an oracle that upon given an index i returns the i-th character y_i of y.

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Probabilistic Checkable Proofs

Definition 107 (PCP)
A language $L \in \mathrm{PCP}_{c(n),s(n)}(r(n), q(n))$ if there exists a polynomial time, non-adaptive, randomized verifier V, s.t.

[$x \in L$] There exists a proof string y, s.t. $V^{\pi_y}(x) =$ "accept" with proability $\geq c(n)$.

[$x \notin L$] For any proof string y, $V^{\pi_y}(x) =$ "accept" with probability $\leq s(n)$.

The verifier uses at most $\mathcal{O}(r(n))$ random bits and makes at most $\mathcal{O}(q(n))$ oracle queries.

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Probabilistic Checkable Proofs

$c(n)$ is called the completeness. If not specified otw. $c(n) = 1$. Probability of accepting a correct proof.

$s(n) < c(n)$ is called the soundness. If not specified otw. $s(n) = 1/2$. Probability of accepting a wrong proof.

$r(n)$ is called the randomness complexity, i.e., how many random bits the (randomized) verifier uses.

$q(n)$ is the query complexity of the verifier.

Probabilistic Checkable Proofs

Definition 107 (PCP)
A language $L \in \mathrm{PCP}_{c(n),s(n)}(r(n), q(n))$ if there exists a polynomial time, non-adaptive, randomized verifier V, s.t.

[$x \in L$] There exists a proof string y, s.t. $V^{\pi_y}(x) =$ "accept" with proability $\geq c(n)$.

[$x \notin L$] For any proof string y, $V^{\pi_y}(x) =$ "accept" with probability $\leq s(n)$.

The verifier uses at most $\mathcal{O}(r(n))$ random bits and makes at most $\mathcal{O}(q(n))$ oracle queries.

Probabilistic Checkable Proofs

- $P = PCP(0, 0)$
  verifier without randomness and proof access is deterministic algorithm
- $PCP(\log n, 0) \subseteq P$
  we can simulate $O(\log n)$ random bits in deterministic, polynomial time
- $PCP(0, \log n) \subseteq P$
  we can simulate short proofs in polynomial time
- $PCP(\text{poly}(n), 0) = coRP \stackrel{?!}{=} P$
  by definition; coRP is randomized polytime with one sided error (positive probability of accepting NO-instance)

Note that the first three statements also hold with equality

Probabilistic Checkable Proofs

$c(n)$ is called the completeness. If not specified otw. $c(n) = 1$. Probability of accepting a correct proof.

$s(n) < c(n)$ is called the soundness. If not specified otw. $s(n) = 1/2$. Probability of accepting a wrong proof.

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- $\mathrm{PCP}(0, \mathrm{poly}(n)) = \mathrm{NP}$
  by definition; NP-verifier does not use randomness and asks polynomially many queries
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PCP theorem: Proof System View

Theorem 108 (PCP Theorem B)
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Probabilistic Proof for Graph NonIsomorphism

GNI is the language of pairs of non-isomorphic graphs

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It expects a proof of the following form:

- For any labeled n-node graph H the H's bit $P[H]$ of the proof fulfills

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G_0 \equiv H &\implies P[H] = 0\\
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Probabilistic Proof for Graph NonIsomorphism

Verifier:

- choose $b \in \{0,1\}$ at random
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Version B $\Longrightarrow$ Version A

- For 3SAT there exists a verifier that uses $c \log n$ random bits, reads $q = \mathcal{O}(1)$ bits from the proof, has completeness 1 and soundness $1/2$.
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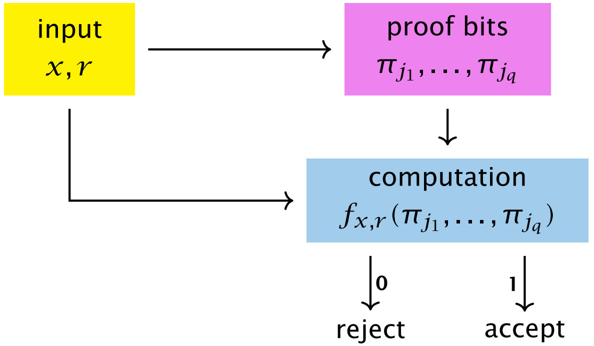

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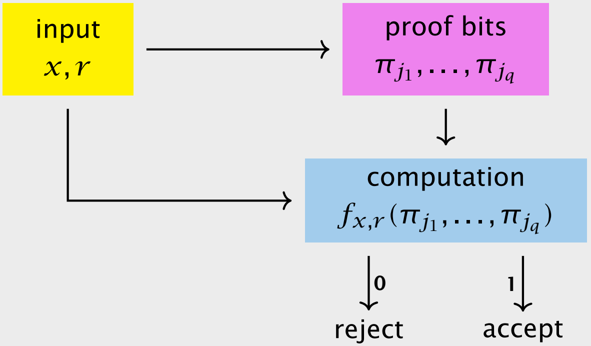

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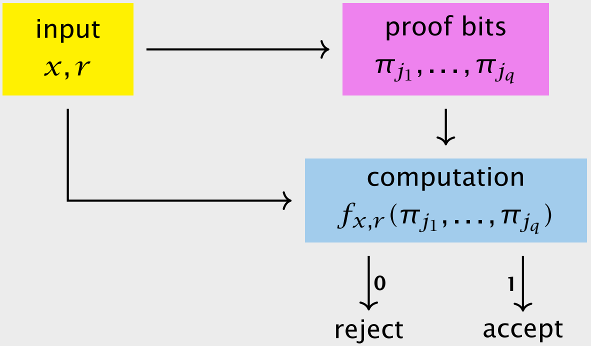

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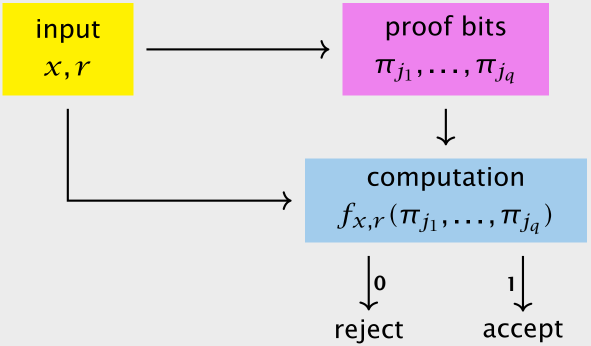

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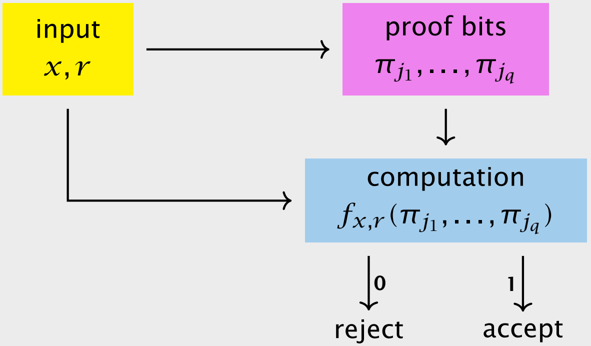

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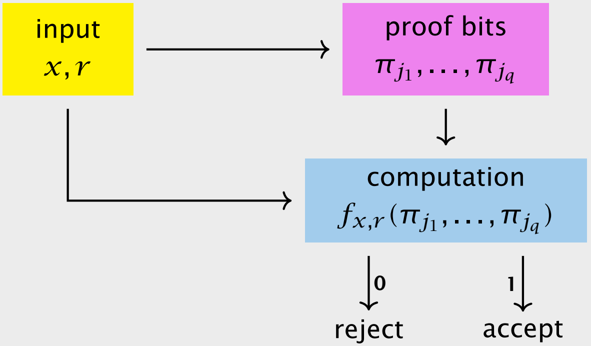

Version A $\Longrightarrow$ Version B

We show: Version A $\Longrightarrow$ $\mathrm{NP} \subseteq \mathrm{PCP}_{1,1-\epsilon}(\log n, 1)$.

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The idea is to encode an NP-witness (e.g. a satisfying assignment (say n bits)) by a code whose code-words have 2^n bits.

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The Code

$u \in \{0,1\}^n$ (satisfying assignment)

Walsh-Hadamard Code:
$\mathrm{WH}_u : \{0,1\}^n \to \{0,1\}, x \mapsto x^T u$ (over $\mathrm{GF}(2)$)

The code-word for u is WH_u. We identify this function by a bit-vector of length 2^n.

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Lemma 109
If $u \neq u'$ then WH_u and $\mathrm{WH}_{u'}$ differ in at least 2^{n-1} bits.

Proof:
Suppose that $u - u' \neq 0$. Then

$$\mathrm{WH}_u(x) \neq \mathrm{WH}_{u'}(x) \iff (u - u')^T x \neq 0$$

This holds for 2^{n-1} different vectors x.

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Since the set of codewords is the set of all linear functions $\{0,1\}^n$ to $\{0,1\}$ we can check

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Let $\rho \in [0,1]$. We say that $f, g : \{0,1\}^n \to \{0,1\}$ are ρ-close if

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Theorem 111 (proof deferred)
Let $f : \{0,1\}^n \to \{0,1\}$ with

$$\Pr_{x,y \in \{0,1\}^n}\Big[f(x) + f(y) = f(x+y)\Big] \geq \rho > \frac{1}{2} \ .$$

Then there is a linear function $\tilde{f}$ such that f and $\tilde{f}$ are ρ-close.

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Suppose for $\delta < 1/4$ f is $(1-\delta)$-close to some linear function $\tilde{f}$.

$\tilde{f}$ is uniquely defined by f, since linear functions differ on at least half their inputs.

Suppose we are given $x \in \{0,1\}^n$ and access to f. Can we compute $\tilde{f}(x)$ using only constant number of queries?

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1. Choose $x' \in \{0,1\}^n$ u.a.r.
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4. Output $y' + y''$.

x' and x'' are uniformly distributed (albeit dependent). With probability at least $1 - 2\delta$ we have $f(x') = \tilde{f}(x')$ and $f(x'') = \tilde{f}(x'')$.

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This technique is known as local decoding of the Walsh-Hadamard code.

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We show that $\text{QUADEQ} \in \text{PCP}(\text{poly}(n), 1)$. The theorem follows since any PCP-class is closed under polynomial time reductions.

QUADEQ
Given a system of quadratic equations over $\text{GF}(2)$. Is there a solution?

$\text{NP} \subseteq \text{PCP}(\text{poly}(n), 1)$

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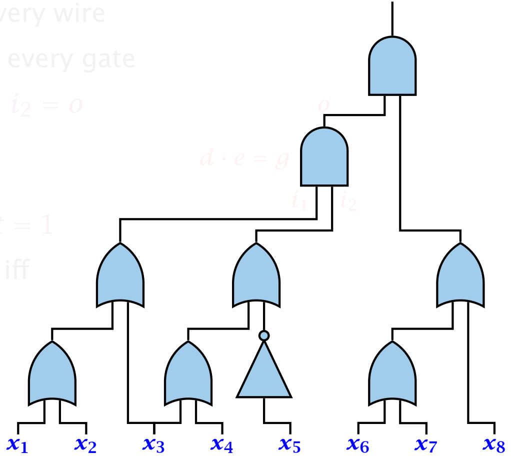
QUADEQ is NP-complete

- given 3SAT instance C represent it as Boolean circuit
  e.g. $C = (x_1 \lor x_2 \lor x_3) \land (x_3 \lor x_4 \lor \bar{x}_5) \land (x_6 \lor x_7 \lor x_8)$
- add variable for every wire
- add constraint for every gate

  OR: $i_1 + i_2 + i_1 \cdot i_2 = o$

  AND: $i_1 \cdot i_2 = o$

  NEG: $i = 1 - o$
- add constraint $out = 1$
- system is feasible iff C is satisfiable



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We show that $\text{QUADEQ} \in \text{PCP}(\text{poly}(n), 1)$. The theorem follows since any PCP-class is closed under polynomial time reductions.

QUADEQ
Given a system of quadratic equations over $\text{GF}(2)$. Is there a solution?

QUADEQ is NP-complete

- given 3SAT instance C represent it as Boolean circuit
  e.g. $C = (x_1 \vee x_2 \vee x_3) \wedge (x_3 \vee x_4 \vee \bar{x}_5) \wedge (x_6 \vee x_7 \vee x_8)$
- add variable for every wire
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  OR: $i_1 + i_2 + i_1 \cdot i_2 = o$

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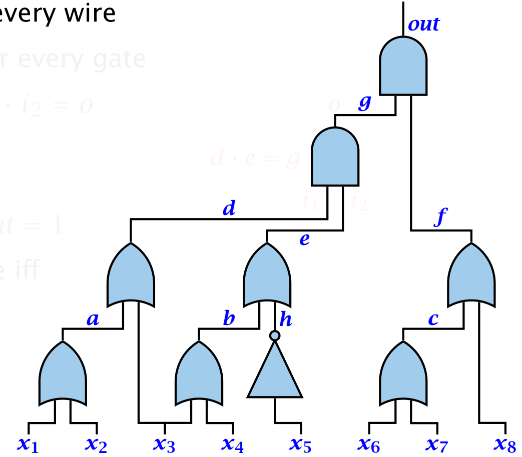

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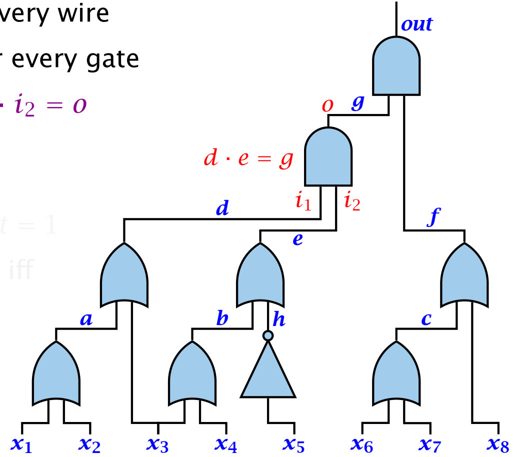

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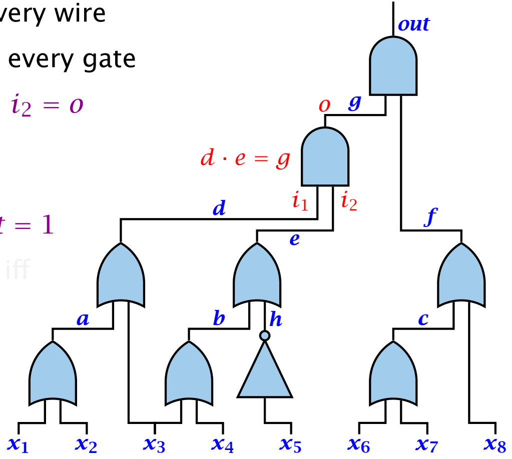

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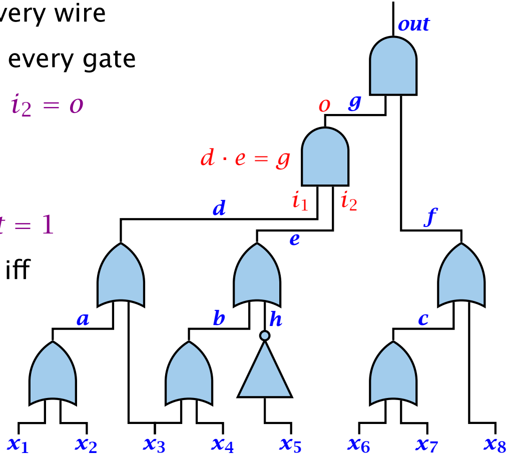

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We encode an instance of QUADEQ by a matrix A that has n^2 columns; one for every pair i, j; and a right hand side vector b.

For an n-dimensional vector x we use $x \otimes x$ to denote the n^2-dimensional vector whose i, j-th entry is $x_i x_j$.

Then we are asked whether

$$A(x \otimes x) = b$$

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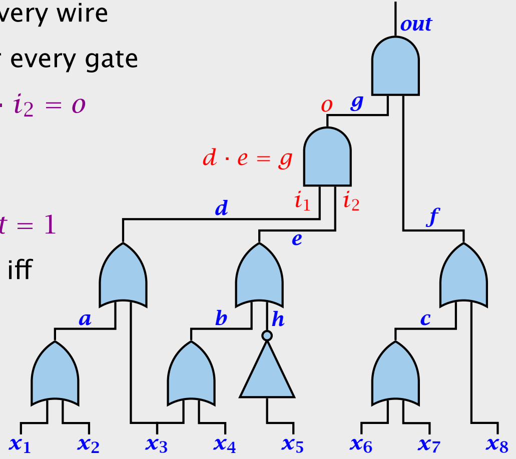

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Step 1. Linearity Test.
The proof contains $2^n + 2^{n^2}$ bits. This is interpreted as a pair of functions $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^{n^2} \to \{0,1\}$.

We do a 0.999-linearity test for both functions (requires a constant number of queries).

We also assume that for the remaining constant number of accesses WH-decoding succeeds and we recover $\tilde{f}(x)$.

Hence, our proof will only ever see $\tilde{f}$. To simplify notation we use f for $\tilde{f}$, in the following (similar for g, $\tilde{g}$).

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Step 2. Verify that g encodes $u \otimes u$ where u is string encoded by f.

$f(r) = u^T r$ and $g(z) = w^T z$ since f, g are linear.

- choose r, r' independently, u.a.r. from $\{0,1\}^n$
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- repeat 3 times

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If $U \neq W$ then $Wr' \neq Ur'$ with probability at least $1/2$. Then $r^T W r' \neq r^T U r'$ with probability at least $1/4$.

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Step 3. Verify that f encodes satisfying assignment.

We need to check

$$A_k(u \otimes u) = b_k$$

where A_k is the k-th row of the constraint matrix. But the left hand side is just $g(A_k^T)$.

We can handle this by a single query but checking all constraints would take $\mathcal{O}(m)$ steps.

We compute $r^T A$, where $r \in_R \{0,1\}^m$. If u is not a satisfying assignment then with probability 1/2 the vector r will hit an odd number of violated constraints.

In this case $r^T A(u \otimes u) \neq r^T b_k$. The left hand side is equal to $g(A^T r)$.

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Let W be $n \times n$-matrix with entries from w. Let U be matrix with $U_{ij} = u_i \cdot u_j$ (entries from $u \otimes u$).

$$g(r \otimes r') = w^T(r \otimes r') = \sum_{ij} w_{ij} r_i r'_j = r^T W r'$$

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If $U \neq W$ then $Wr' \neq Ur'$ with probability at least 1/2. Then $r^T W r' \neq r^T U r'$ with probability at least 1/4.

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Step 3. Verify that f encodes satisfying assignment.

We need to check

$$A_k(u \otimes u) = b_k$$

where A_k is the k-th row of the constraint matrix. But the left hand side is just $g(A_k^T)$.

We can handle this by a single query but checking all constraints would take $\mathcal{O}(m)$ steps.

We compute $r^T A$, where $r \in_R \{0,1\}^m$. If u is not a satisfying assignment then with probability 1/2 the vector r will hit an odd number of violated constraints.

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We used the following theorem for the linearity test:

Theorem 111
Let $f : \{0,1\}^n \to \{0,1\}$ with

$$\Pr_{x,y\in\{0,1\}^n}\Big[f(x) + f(y) = f(x+y)\Big] \geq \rho > \frac{1}{2} \ .$$

Then there is a linear function $\tilde{f}$ such that f and $\tilde{f}$ are ρ-close.

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Fourier Transform over $\mathrm{GF}(2)$

In the following we use $\{-1, 1\}$ instead of $\{0, 1\}$. We map $b \in \{0, 1\}$ to $(-1)^b$.

This turns summation into multiplication.

The set of function $f : \{-1, 1\}^n \to \mathbb{R}$ form a 2^n-dimensional Hilbert space.

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Hilbert space

- addition $(f+g)(x) = f(x) + g(x)$
- scalar multiplication $(\alpha f)(x) = \alpha f(x)$
- inner product $\langle f, g\rangle = E_{x\in\{-1,1\}^n}[f(x)g(x)]$ (bilinear, $\langle f, f\rangle \geq 0$, and $\langle f, f\rangle = 0 \Rightarrow f = 0$)
- completeness: any sequence x_k of vectors for which

$$\sum_{k=1}^{\infty} \|x_k\| < \infty \text{ fulfills } \left\| L - \sum_{k=1}^{N} x_k \right\| \to 0$$

for some vector L.

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standard basis

$$e_x(y) = \begin{cases} 1 & x = y \\ 0 & \text{otw.} \end{cases}$$

Then, $f(x) = \sum_i \alpha_i e_i(x)$ where $\alpha_x = f(x)$, this means the functions e_i form a basis. This basis is orthonormal.

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A function χ_α multiplies a set of x_i's. Back in the $\mathrm{GF}(2)$-world this means summing a set of z_i's where $x_i = (-1)^{z_i}$.

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We can write any function $f : \{-1,1\}^n \to \mathbb{R}$ as

$$f = \sum_\alpha \hat{f}_\alpha \chi_\alpha$$

We call $\hat{f}_\alpha$ the α^{th} Fourier coefficient.

Lemma 112

1. $\langle f, g\rangle = \sum_\alpha f_\alpha g_\alpha$
2. $\langle f, f\rangle = \sum_\alpha f_\alpha^2$

Note that for Boolean functions $f : \{-1,1\}^n \to \{-1,1\}$, $\langle f, f\rangle = 1$.

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Linearity Test

in GF(2):
We want to show that if $\Pr_{x,y}[f(x)+f(y)=f(x+y)]$ is large than f has a large agreement with a linear function.

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&= \sum\nolimits_{\alpha,\beta,\gamma} \hat{f}_\alpha \hat{f}_\beta \hat{f}_\gamma \cdot E_x\left[\chi_\alpha(x)\chi_\beta(x)\right] E_y\left[\chi_\alpha(y)\chi_\gamma(y)\right] \\
&= \sum_\alpha \hat{f}_\alpha^3
\end{aligned}
$$

Linearity Test

$$
\Pr_{x,y}[f(x\circ y) = f(x)f(y)] \geq \frac{1}{2} + \epsilon
$$

means that the fraction of inputs x, y on which $f(x\circ y)$ and $f(x)f(y)$ agree is at least $1/2+\epsilon$.

This gives

$$
\begin{aligned}
E_{x,y}[f(x\circ y)f(x)f(y)] &= \text{agreement} - \text{disagreement} \\
&= 2\text{agreement} - 1 \\
&\geq 2\epsilon
\end{aligned}
$$

$$
\begin{aligned}
2\epsilon &\le E_{x,y}\Big[f(x\circ y)f(x)f(y)\Big] \\
&= E_{x,y}\Big[\Big(\textstyle\sum_\alpha \hat{f}_\alpha\chi_\alpha(x\circ y)\Big)\cdot\Big(\sum_\beta \hat{f}_\beta\chi_\beta(x)\Big)\cdot\Big(\sum_\gamma \hat{f}_\gamma\chi_\gamma(y)\Big)\Big] \\
&= E_{x,y}\Big[\textstyle\sum_{\alpha,\beta,\gamma}\hat{f}_\alpha\hat{f}_\beta\hat{f}_\gamma\chi_\alpha(x)\chi_\alpha(y)\chi_\beta(x)\chi_\gamma(y)\Big] \\
&= \textstyle\sum_{\alpha,\beta,\gamma}\hat{f}_\alpha\hat{f}_\beta\hat{f}_\gamma\cdot E_x\Big[\chi_\alpha(x)\chi_\beta(x)\Big]E_y\Big[\chi_\alpha(y)\chi_\gamma(y)\Big] \\
&= \sum_\alpha \hat{f}_\alpha^3 \\
&\le \max_\alpha \hat{f}_\alpha\cdot\sum_\alpha \hat{f}_\alpha^2 = \max_\alpha \hat{f}_\alpha
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Approximation Preserving Reductions

AP-reduction

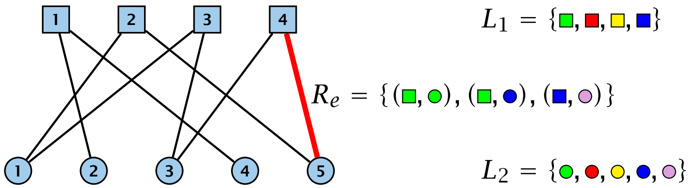
- $x \in I_1 \Rightarrow f(x,r) \in I_2$
- $\mathrm{SOL}_1(x) \neq \emptyset \Rightarrow \mathrm{SOL}_1(f(x,r)) \neq \emptyset$
- $y \in \mathrm{SOL}_2(f(x,r)) \Rightarrow g(x,y,r) \in \mathrm{SOL}_1(x)$
- f, g are polynomial time computable
- $R_2(f(x,r),y) \leq r \Rightarrow R_1(x,g(x,y,r)) \leq 1 + \alpha(r-1)$

$$
\begin{aligned}
2\epsilon &\leq E_{x,y}\Big[f(x \circ y)f(x)f(y)\Big]\\
&= E_{x,y}\Big[\big(\textstyle\sum_\alpha \hat{f}_\alpha \chi_\alpha(x \circ y)\big) \cdot \big(\sum_\beta \hat{f}_\beta \chi_\beta(x)\big) \cdot \big(\sum_\gamma \hat{f}_\gamma \chi_\gamma(y)\big)\Big]\\
&= E_{x,y}\Big[\textstyle\sum_{\alpha,\beta,\gamma} \hat{f}_\alpha \hat{f}_\beta \hat{f}_\gamma \chi_\alpha(x)\chi_\alpha(y)\chi_\beta(x)\chi_\gamma(y)\Big]\\
&= \textstyle\sum_{\alpha,\beta,\gamma} \hat{f}_\alpha \hat{f}_\beta \hat{f}_\gamma \cdot E_x\Big[\chi_\alpha(x)\chi_\beta(x)\Big] E_y\Big[\chi_\alpha(y)\chi_\gamma(y)\Big]\\
&= \sum_\alpha \hat{f}_\alpha^3\\
&\leq \max_\alpha \hat{f}_\alpha \cdot \sum_\alpha \hat{f}_\alpha^2 = \max_\alpha \hat{f}_\alpha
\end{aligned}
$$

Label Cover

Input:

- bipartite graph $G = (V_1, V_2, E)$
- label sets L_1, L_2
- for every edge $(u,v) \in E$ a relation $R_{u,v} \subseteq L_1 \times L_2$ that describe assignments that make the edge happy.
- maximize number of happy edges



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Label Cover

- an instance of label cover is (d_1, d_2)-regular if every vertex in L_1 has degree d_1 and every vertex in L_2 has degree d_2.
- if every vertex has the same degree d the instance is called d-regular

Minimization version:

- assign a set $L_x \subseteq L_1$ of labels to every node $x \in L_1$ and a set $L_y \subseteq L_2$ to every node $y \in L_2$
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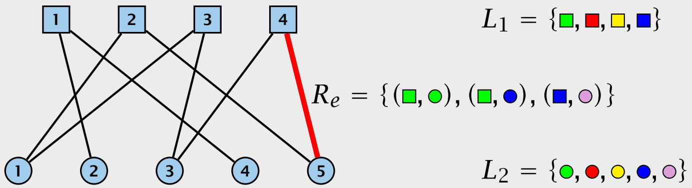

MAX E3SAT via Label Cover

instance:

$\Phi(x) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_4 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_4)$

corresponding graph:

label sets: $L_1 = \{T, F\}^3, L_2 = \{T, F\}$ (T=true, F=false)

relation: $R_{C,x_i} = \{((u_i, u_j, u_k), u_i)\}$, where the clause C is over variables x_i, x_j, x_k and assignment (u_i, u_j, u_k) satisfies C

$$R = \{((F,F,F),F), ((F,T,F),F), ((F,F,T),T), ((F,T,T),T), ((T,T,T),T), ((T,T,F),F), ((T,F,F),F)\}$$

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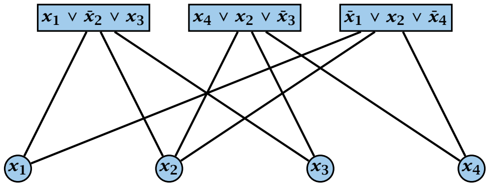

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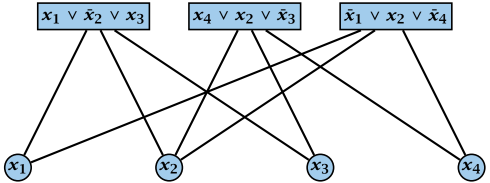

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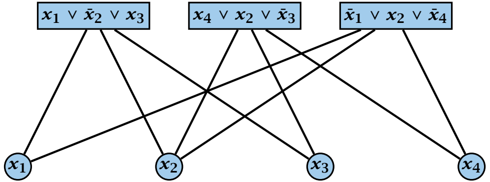

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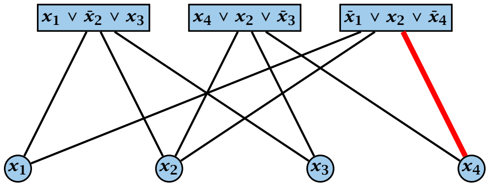

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If we can satisfy k out of m clauses in ϕ we can make at least $3k + 2(m - k)$ edges happy.

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MAX E3SAT via Label Cover

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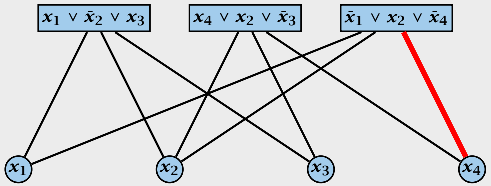

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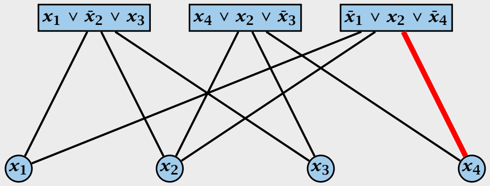

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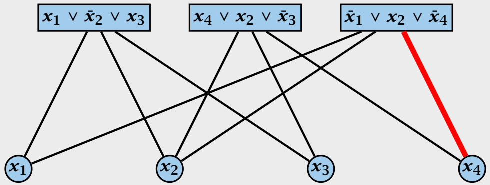

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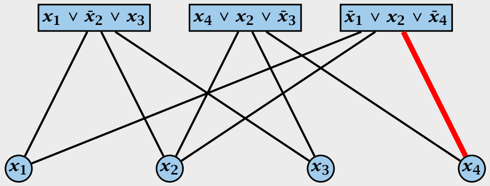

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Lemma 114
If we can satisfy at most k *clauses in* Φ *we can make at most* $3k + 2(m - k) = 2m + k$ *edges happy.*

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Lemma 114
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Proof:

- the labeling of nodes in V_2 gives an assignment
- every unsatisfied clause in this assignment cannot be assigned a label that satisfies all 3 incident edges
- hence at most $3m - (m - k) = 2m + k$ edges are happy

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Hardness for Label Cover

We cannot distinguish between the following two cases

- all $3m$ edges can be made happy
- at most $2m + (1-\epsilon)m = (3-\epsilon)m$ out of the $3m$ edges can be made happy

Hence, we cannot obtain an approximation constant $\alpha > \frac{3-\epsilon}{3}$.

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$(3,5)$-regular instances

Theorem 115

There is a constant ρ s.t. MAXE3SAT is hard to approximate with a factor of ρ even if restricted to instances where a variable appears in exactly 5 clauses.

Then our reduction has the following properties:

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Hardness for Label Cover

We cannot distinguish between the following two cases

- all $3m$ edges can be made happy
- at most $2m + (1-\epsilon)m = (3-\epsilon)m$ out of the $3m$ edges can be made happy

Hence, we cannot obtain an approximation constant $\alpha > \frac{3-\epsilon}{3}$.

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Here MAX3SAT(≤ 29) is the variant of MAX3SAT in which a variable appears in at most 29 clauses. Similar for the other problems.

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If I has the uniqueness property than also I'.

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Non interactive agreement:

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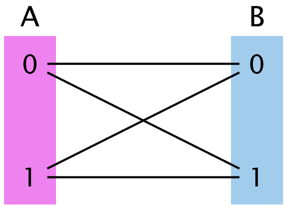

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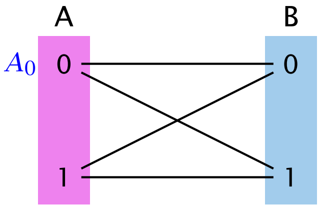

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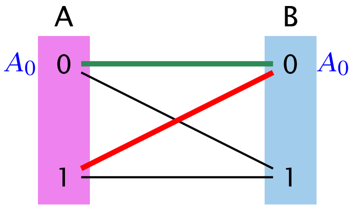

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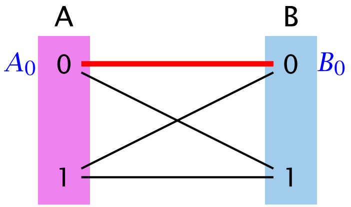

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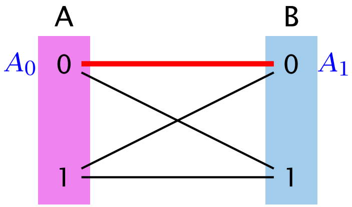

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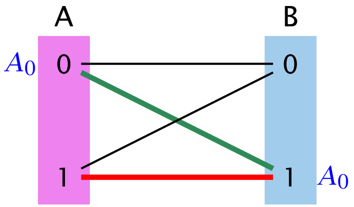

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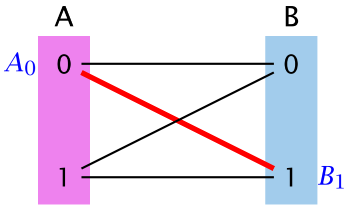

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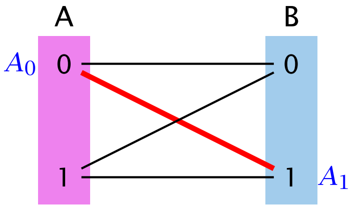

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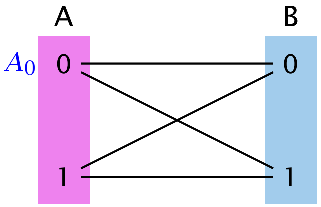

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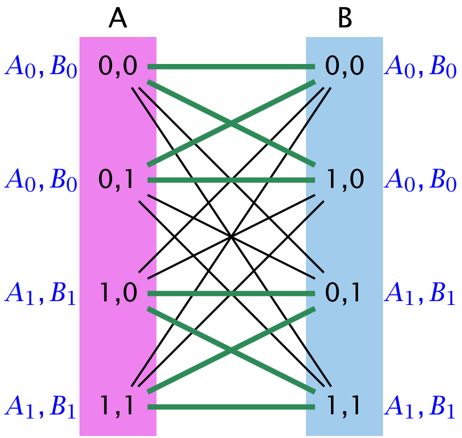

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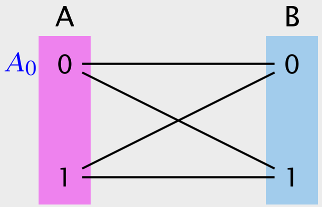

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Boosting

Theorem 117
There is a constant $c > 0$ *such if* $\mathrm{OPT}(I) = |E|(1-\delta)$ *then* $\mathrm{OPT}(I') \leq |E'|(1-\delta)^{\frac{ck}{\log L}}$, *where* $L = |L_1| + |L_2|$ *denotes total number of labels in* I.

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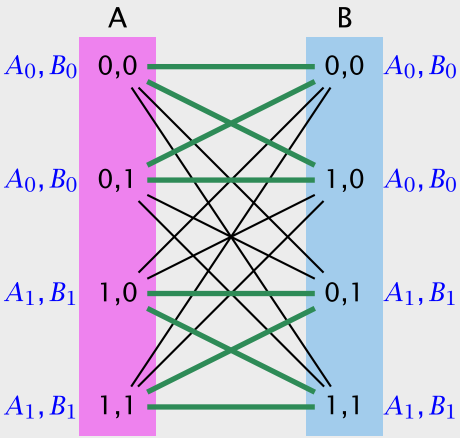

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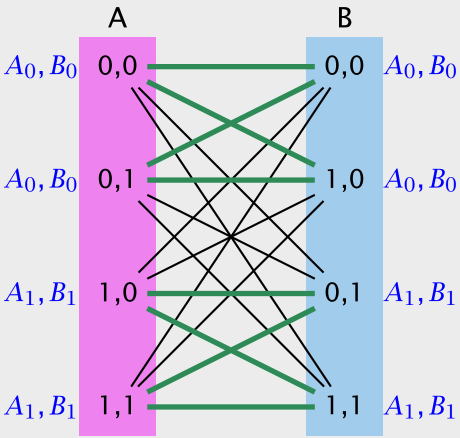

Hardness of Label Cover

Theorem 118

There are constants $c > 0$, $\delta < 1$ s.t. for any k we cannot distinguish regular instances for Label Cover in which either

- $\mathrm{OPT}(I) = |E|$, *or*
- $\mathrm{OPT}(I) = |E|(1-\delta)^{ck}$

unless each problem in NP has an algorithm running in time $\mathcal{O}(n^{\mathcal{O}(k)})$.

Corollary 119

There is no α-approximation for Label Cover for any constant α.

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Hardness of Set Cover

Theorem 120
There exist regular Label Cover instances s.t. we cannot distinguish whether

- *all edges are satisfiable, or*
- *at most a* $1/\log^2(|L_1||E|)$*-fraction is satisfiable*

unless NP-problems have algorithms with running time $\mathcal{O}(n^{\mathcal{O}(\log\log n)})$.

choose $k \geq \frac{2}{c}\log_{1/(1-\delta)}(\log(|L_1||E|)) = \mathcal{O}(\log\log n)$.

Hardness of Label Cover

Theorem 118
There are constants $c > 0$, $\delta < 1$ *s.t. for any* k *we cannot distinguish regular instances for Label Cover in which either*

- $\mathrm{OPT}(I) = |E|$, *or*
- $\mathrm{OPT}(I) = |E|(1-\delta)^{ck}$

unless each problem in NP has an algorithm running in time $\mathcal{O}(n^{\mathcal{O}(k)})$.

Corollary 119
There is no α*-approximation for Label Cover for any constant* α.

Hardness of Set Cover

Partition System (s, t, h)

- universe U of size s
- t pairs of sets $(A_1, \bar{A}_1), \dots, (A_t, \bar{A}_t)$;
  $A_i \subseteq U, \bar{A}_i = U \setminus A_i$
- choosing from any h pairs only one of A_i, $\bar{A}_i$ we do not cover the whole set U

we will show later:
for any h, t with $h \le t$ there exist systems with $s = |U| \le 4t^2 2^h$

Hardness of Set Cover

Theorem 120
There exist regular Label Cover instances s.t. we cannot distinguish whether

- *all edges are satisfiable, or*
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unless NP-problems have algorithms with running time $\mathcal{O}(n^{\mathcal{O}(\log\log n)})$.

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Hardness of Set Cover

Given a Label Cover instance we construct a Set Cover instance;

The universe is $E \times U$, where U is the universe of some partition system; ($t = |L_1|$, $h = \log(|E||L_1|)$)

for all $u \in V_1, \ell_1 \in L_1$

$$S_{u,\ell_1} = \{((u,v),a) \mid (u,v) \in E, a \in A_{\ell_1}\}$$

for all $v \in V_2, \ell_2 \in L_2$

$$S_{v,\ell_2} = \{((u,v),a) \mid (u,v) \in E, a \in \bar{A}_{\ell_1}, \text{ where } (\ell_1,\ell_2) \in R_{(u,v)}\}$$

note that S_{v,ℓ_2} is well defined because of uniqueness property

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Hardness of Set Cover

Suppose that we can make all edges happy.

Choose sets S_{u,ℓ_1}'s and S_{v,ℓ_2}'s, where ℓ_1 is the label we assigned to u, and ℓ_2 the label for v. ($|V_1|+|V_2|$ sets)

For an edge (u,v), S_{v,ℓ_2} contains $\{(u,v)\} \times A_{\ell_2}$. For a happy edge S_{u,ℓ_1} contains $\{(u,v)\} \times \bar{A}_{\ell_2}$.

Since all edges are happy we have covered the whole universe.

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Lemma 121

Given a solution to the set cover instance using at most $\frac{h}{8}(|V_1|+|V_2|)$ sets we can find a solution to the Label Cover instance satisfying at least $\frac{2}{h^2}|E|$ edges.

If the Label Cover instance cannot satisfy a $2/h^2$-fraction we cannot cover with $\frac{h}{8}(|V_1|+|V_2|)$ sets.

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Hardness of Set Cover

- n_u: number of $S_{u,i}$'s in cover
- n_v: number of $S_{v,j}$'s in cover
- at most $1/4$ of the vertices can have $n_u, n_v \ge h/2$; mark these vertices
- at least half of the edges have both end-points unmarked, as the graph is regular
- for such an edge (u,v) we must have chosen $S_{u,i}$ and a corresponding $S_{v,j}$, s.t. $(i,j) \in R_{u,v}$ (making (u,v) happy)
- we choose a random label for u from the (at most $h/2$) chosen $S_{u,i}$-sets and a random label for v from the (at most $h/2$) $S_{v,j}$-sets
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- we choose a random label for u from the (at most $h/2$) chosen $S_{u,i}$-sets and a random label for v from the (at most $h/2$) $S_{v,j}$-sets
- (u, v) gets happy with probability at least $4/h^2$
- hence we make a $2/h^2$-fraction of edges happy

Hardness of Set Cover

Lemma 121

Given a solution to the set cover instance using at most $\frac{h}{8}(|V_1| + |V_2|)$ sets we can find a solution to the Label Cover instance satisfying at least $\frac{2}{h^2}|E|$ edges.

If the Label Cover instance cannot satisfy a $2/h^2$-fraction we cannot cover with $\frac{h}{8}(|V_1| + |V_2|)$ sets.

Since differentiating between both cases for the Label Cover instance is hard, we have an $\mathcal{O}(h)$-hardness for Set Cover.

Hardness of Set Cover

- n_u: number of $S_{u,i}$'s in cover
- n_v: number of $S_{v,j}$'s in cover
- at most 1/4 of the vertices can have $n_u, n_v \geq h/2$; mark these vertices
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Theorem 122

There is no $\frac{1}{32}\log n$-approximation for the unweighted Set Cover problem unless problems in NP can be solved in time $\mathcal{O}(n^{\mathcal{O}(\log\log n)})$.

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Set $h = \log(|E||L_1|)$ and $t = |L_1|$; Size of partition system is

$$s = |U| = 4t^2 2^h = 4|L_1|^2(|E||L_1|)^2 = 4|E|^2|L_1|^4$$

The size of the ground set is then

$$n = |E||U| = 4|E|^3|L_2|^4 \le (|E||L_2|)^4$$

for sufficiently large $|E|$. Then $h \ge \frac{1}{4}\log n$.

If we get an instance where all edges are satisfiable there exists a cover of size only $|V_1| + |V_2|$.

If we find a cover of size at most $\frac{h}{8}(|V_1| + |V_2|)$ we can use this to satisfy at least a fraction of $2/h^2 \ge 1/\log^2(|E||L_1|)$ of the edges. this is not possible...

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What is the probability that a given choice covers U?

The probability that an element $u \in A_i$ is $1/2$ (same for $\bar{A}_i$).

The probability that u is covered is $1 - \frac{1}{2^h}$.

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The random process outputs a partition system with constant probability!

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Advanced PCP Theorem

Theorem 124
For any positive constant $\epsilon > 0$, it is the case that $\mathrm{NP} \subseteq \mathrm{PCP}_{1-\epsilon,1/2+\epsilon}(\log n, 3)$. Moreover, the verifier just reads three bits from the proof, and bases its decision only on the parity of these bits.

It is NP-hard to approximate a MAXE3LIN problem by a factor better than $1/2 + \delta$, for any constant δ.

It is NP-hard to approximate MAX3SAT better than $7/8 + \delta$, for any constant δ.

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