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Randomized Algorithms

Exercise Sheet 7

Due: December 7, 2015
at 10:15, in class

Exercise 7.1 (10 points)
Suppose that we have an algorithm that takes as input a string of n bits. We are told that the expected running time is $O(n^2)$ if the input bits are chosen independently and uniformly at random. What can Markov's inequality tell us about the worst-case running time of this algorithm on inputs of size n?

Exercise 7.2 (10 points)
Consider a non-negative random variable X with expectation μ_X and variance σ_X^2.

(a) Show that for any $t \in \mathbb{R}^+$,

$$P[X - \mu_X \geq t\sigma_X] \leq \frac{1}{1+t^2}$$

(b) Prove that

$$P[|X - \mu_X| \geq t\sigma_X] \leq \frac{2}{1+t^2}$$

(c) Under what circumstances does the inequality of (b) give a better bound than Chebyshev's inequality?

Exercise 7.3 (10 points)
We flip a fair coin n times to obtain n random bits $X_1, \ldots, X_n \in \{0,1\}$. We look at all $m = \binom{n}{2}$ pairs of random bits in a fixed order and define for this $Y_i := X_j \oplus X_k$, where $\oplus$ is the exclusive-or operator. We further define $Y := \sum_i Y_i$.

Using Chebyshev's inequality, prove a bound on $P\left[|Y - E[Y]| \geq t\right]$.

Exercise 7.4 (10 points)
Generalize the median-finding algorithm to find the kth largest item in a set of n items for any given value of k. Prove that your resulting algorithm is correct, and bound its running time.
Hint: Show where the code of the Randomized Median Algorithm should be changed and show which parts of the proof are affected. Also show that the probabilities for the "bad events" can be bounded by the same values as in the proof of the Randomized Median Algorithm.