### **Definition 1**

A 0-1 sequence S is bitonic if it can be written as the concatenation of subsequences  $S_1$  and  $S_2$  such that either

- $ightharpoonup S_1$  is monotonically increasing and  $S_2$  monotonically decreasing, or
- $S_1$  is monotonically decreasing and  $S_2$  monotonically increasing.

Note, that this just defines bitonic 0-1 sequences. Bitonic sequences are defined differently.



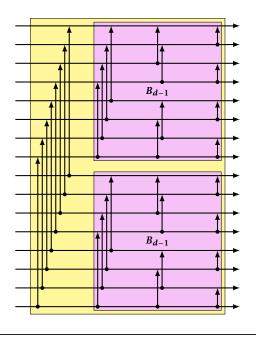
## **Bitonic Merger**

### Bitonic Merger $B_d$

The bitonic merger  $B_d$  of dimension d is constructed by combining two bitonic mergers of dimension d-1.

If we feed a bitonic 0-1 sequence into this, the sequence will be sorted.

(actually, any bitonic sequence will be sorted, but we do not prove this)



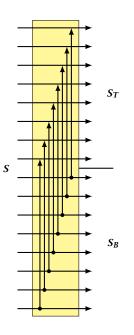
## **Bitonic Merger**

If we feed a bitonic 0-1 sequence S into the network on the right we obtain two bitonic sequences  $S_T$  and  $S_B$  s.t.

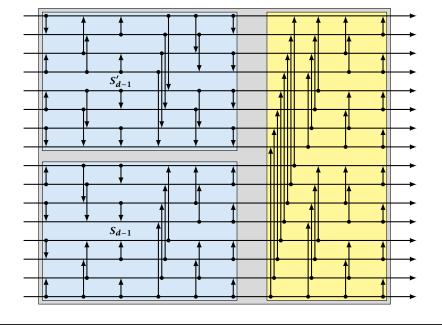
- 1.  $S_B \leq S_T$  (element-wise)
- **2.**  $S_B$  and  $S_T$  are bitonic

#### **Proof:**

- ▶ assume wlog. *S* more 1's than 0's.
- ▶ assume for contradiction two 0s at same comparator  $(i, j = i + 2^d)$ 
  - everything 0 btw i and j means we have more than 50% zeros ( $\xi$ ).
  - ▶ all 1s btw. i and j means we have less than 50% ones ( $\xi$ ).
  - ▶ 1 btw. *i* and *j* and elsewhere means *S* is not bitonic (﴿).



# Bitonic Sorter S<sub>d</sub>



Bitonic Merger:  $(n = 2^d)$ 

▶ comparators:  $C(n) = 2C(n/2) + n/2 \Rightarrow C(n) = O(n \log n)$ .

▶ depth:  $D(n) = D(n/2) + 1 \Rightarrow D(d) = O(\log n)$ .

Bitonic Sorter:  $(n = 2^d)$ 

► comparators:  $C(n) = 2C(n/2) + \mathcal{O}(n \log n) \Rightarrow$  $C(n) = \mathcal{O}(n \log^2 n).$ 

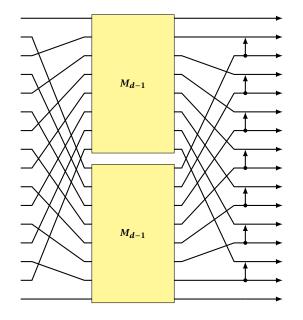
▶ depth:  $D(n) = D(n/2) + \log n \Rightarrow D(n) = \Theta(\log^2 n)$ .

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# **Odd-Even Merge**



## **Odd-Even Merge**

How to merge two sorted sequences?

$$A = (a_1, a_2, ..., a_n), B = (b_1, b_2, ..., b_n), n \text{ even.}$$

Split into odd and even sequences:

$$A_{\text{odd}} = (a_1, a_3, a_5, \dots, a_{n-1}), A_{\text{even}} = (a_2, a_4, a_6, \dots a_n)$$
  
 $B_{\text{odd}} = (b_1, b_3, b_5, \dots, b_{n-1}), B_{\text{even}} = (b_2, b_4, b_6, \dots, b_n)$ 

Let

$$X = merge(A_{odd}, B_{odd})$$
 and  $Y = merge(A_{even}, B_{even})$ 

Then

$$S = (x_1, \min\{x_2, y_1\}, \max\{x_2, y_1\}, \min\{x_3, y_2\}, \dots, y_n)$$

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### **Theorem 2**

There exists a sorting network with depth  $\mathcal{O}(\log n)$  and  $\mathcal{O}(n\log n)$  comparators.