Part II

Linear Programming

Brewery Problem

$\bar{U}$ **Brewery brews ale and beer.**

- Production limited by supply of corn, hops and barley malt
- Recipes for ale and beer require different amounts of resources

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

Brewery Problem

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

How can brewer maximize profits?

- only brew ale: 34 barrels of ale $\Longrightarrow$ 442 €
- only brew beer: 32 barrels of beer $\Longrightarrow$ 736 €
- 7.5 barrels ale, 29.5 barrels beer $\Longrightarrow$ 776 €
- 12 barrels ale, 28 barrels beer $\Longrightarrow$ 800 €

Brewery Problem

Linear Program

- Introduce variables a and b that define how much ale and beer to produce.
- Choose the variables in such a way that the objective function (profit) is maximized.
- Make sure that no constraints (due to limited supply) are violated.

$$\begin{array}{rrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a,b & \ge 0 \end{array}$$

Standard Form LPs

LP in standard form:

- input: numbers a_{ij}, c_j, b_i
- output: numbers x_j
- n = #decision variables, m = #constraints
- maximize linear objective function subject to linear inequalities

$$\begin{array}{llcll} \text{max} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \le i \le m \\ & x_j & \ge & 0 & 1 \le j \le n \end{array}$$

$$\begin{array}{lrcl} \max & c^t x & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

Standard Form LPs

Original LP

$$
\begin{array}{lrcrcl}
\text{max} & 13a & + & 23b & & \\
\text{s.t.} & 5a & + & 15b & \le & 480 \\
 & 4a & + & 4b & \le & 160 \\
 & 35a & + & 20b & \le & 1190 \\
 & & & a,b & \ge & 0
\end{array}
$$

Standard Form

Add a slack variable to every constraint.

$$
\begin{array}{lrcrcrcrcrcl}
\text{max} & 13a & + & 23b & & & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
 & 4a & + & 4b & & & + & s_h & & & = & 160 \\
 & 35a & + & 20b & & & & & + & s_m & = & 1190 \\
 & a & , & b & , & s_c & , & s_h & , & s_m & \ge & 0
\end{array}
$$

Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{lrcl} \max & c^t x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \min & c^t x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

standard maximization form

$$\begin{array}{lrcl} \max & c^t x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

standard minimization form

$$\begin{array}{lrcl} \min & c^t x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **less or equal to equality:**

$$a - 3b + 5c \le 12 \implies \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\ge 0 \end{aligned}$$

- **greater or equal to equality:**

$$a - 3b + 5c \ge 12 \implies \begin{aligned} a - 3b + 5c - s &= 12 \\ s &\ge 0 \end{aligned}$$

- **min to max:**

$$\min a - 3b + 5c \implies \max -a + 3b - 5c$$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **equality to less or equal:**

$$a - 3b + 5c = 12 \implies \begin{array}{rcl} a - 3b + 5c & \leq & 12 \\ -a + 3b - 5c & \leq & -12 \end{array}$$

- **equality to greater or equal:**

$$a - 3b + 5c = 12 \implies \begin{array}{rcl} a - 3b + 5c & \geq & 12 \\ -a + 3b - 5c & \geq & -12 \end{array}$$

- **unrestricted to nonnegative:**

$$x \text{ unrestricted} \implies x = x^+ - x^-,\ x^+ \geq 0,\ x^- \geq 0$$

Standard Form LPs

Observations:

- a linear program does not contain x^2, $\cos(x)$, etc.
- transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

Fundamental Questions

Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^t x \geq \alpha$?

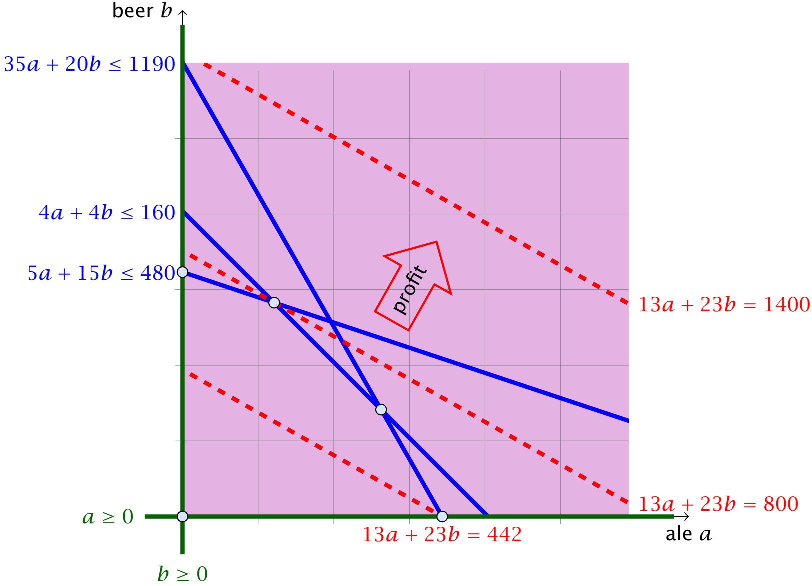
Questions:

- Is LP in NP?
- Is LP in co-NP?
- Is LP in P?

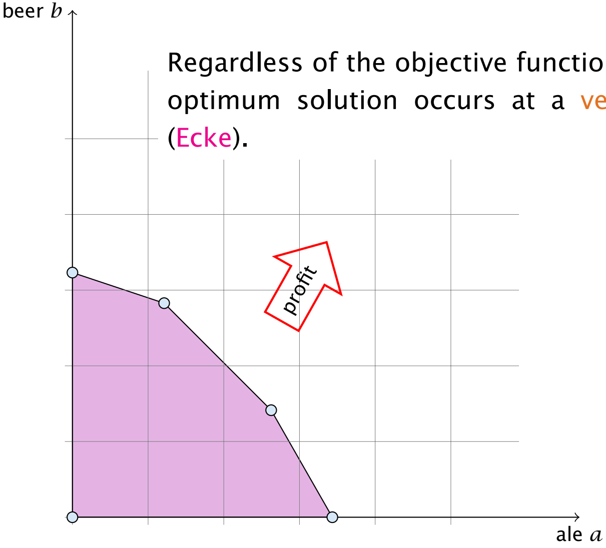
Input size:

- n number of variables, m constraints, L number of bits to encode the input

Geometry of Linear Programming



Geometry of Linear Programming



Regardless of the objective function an optimum solution occurs at a vertex (Ecke).

Convex Sets

A set $S \subseteq \mathbb{R}$ is convex if for all $x, y \in S$ also $\lambda x + (1 - \lambda)y \in S$ for all $0 \leq \lambda \leq 1$.

A point in $x \in S$ that can't be written as a convex combination of two other points in the set is called a vertex.

Definitions

Let for a Linear Program in standard form
$P = \{x \mid Ax = b, x \geq 0\}$.

- P is called the feasible region (Lösungsraum) of the LP.
- A point $x \in P$ is called a feasible point (gültige Lösung).
- If $P \neq \emptyset$ then the LP is called feasible (erfüllbar). Otherwise, it is called infeasible (unerfüllbar).
- An LP is bounded (beschränkt) if it is feasible and
  - $c^t x < \infty$ for all $x \in P$ (for maximization problems)
  - $c^t x > -\infty$ for all $x \in P$ (for minimization problems)

Definitions

Definition 2

A polytop is a set $P \subseteq \mathbb{R}^n$ that is the convex hull of a finite set of points, i.e., $P = \mathrm{conv}(X)$ where

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{\ell} \lambda_i x_i \mid \ell \in \mathbb{N}, x_1, \dots, x_\ell \in X, \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

and $|X| = c$.

Definitions

Definition 3

A polyhedron is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of finitely many half-spaces $\{H(a_1, b_1), \ldots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \left\{x \in \mathbb{R}^n \mid a_i x \leq b_i\right\} \quad .$$

Definitions

Theorem 4

P is a bounded polyhedron iff P is a polytop.

Definition 5

Let $P \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane

$$H(a,b) = \{x \in \mathbb{R}^n \mid ax = b\}$$

is a supporting hyperplane of P if $\max\{ax \mid x \in P\} = b$.

Definition 6

Let $P \subseteq \mathbb{R}^n$. F is a face of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H.

Definition 7

Let $P \subseteq \mathbb{R}^n$.

- v is a vertex of P if $\{v\}$ is a face of P.
- e is an edge of P if e is a face and $\dim(e) = 1$.
- F is a facet of P if F is a face and $\dim(e) = \dim(P) - 1$.

Observation

The feasible region of an LP is a Polyhedron.

Convex Sets

Theorem 8

If there exists an optimal solution to an LP then there exists an optimum solution that is a vertex.

Proof

- suppose x is optimal solution that is not a vertex
- there exists direction $d \neq 0$ such that $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- Wlog. assume $c^t d \geq 0$ (by taking either d or $-d$)
- Consider $x + \lambda d$, $\lambda > 0$

Convex Sets

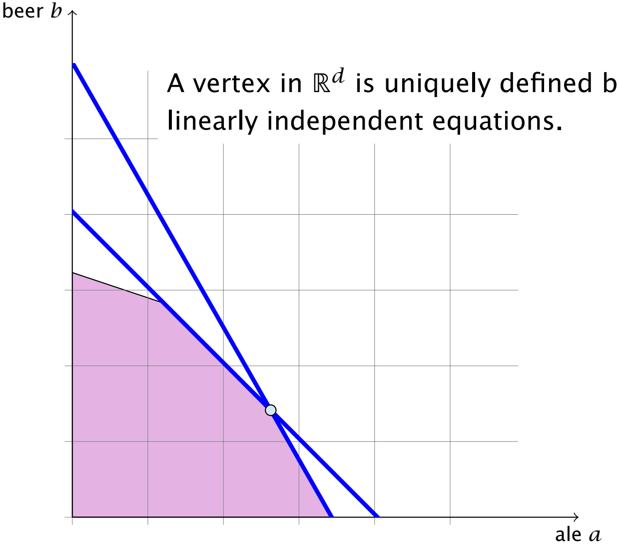
Case 1. $[\exists j \text{ s.t. } d_j < 0]$

- increase λ to λ' until first component of $x + \lambda d$ hits 0
- $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- $c^t x' = c^t(x + \lambda' d) = c^t x + \lambda' c^t d \geq c^t x$

Case 2. $[d_j \geq 0 \text{ for all } j \text{ and } c^t d > 0]$

- $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$
- as $\lambda \to \infty$, $c^t(x + \lambda d) \to \infty$ as $c^t d > 0$

Algebraic View



A vertex in $\mathbb{R}^d$ is uniquely defined by d linearly independent equations.

Notation

Suppose $B \subseteq \{1 \ldots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

Theorem 9

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is a vertex ***iff*** *A_B has linearly independent columns.*

Theorem 9

Let $P = \{x \mid Ax = b, x \geq 0\}$*. For* $x \in P$*, define* $B = \{j \mid x_j > 0\}$*. Then* x *is a vertex* ***iff*** A_B *has linearly independent columns.*

Proof (⇐)

- assume x is not a vertex
- there exists direction d s.t. $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- define $B' = \{j \mid d_j \neq 0\}$
- $A_{B'}$ has linearly dependent columns as $Ad = 0$
- $d_j = 0$ for all j with $x_j = 0$ as $x \pm d \geq 0$
- Hence, $B' \subseteq B$, $A_{B'}$ is sub-matrix of A_B

Theorem 9

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is a vertex **iff** A_B has linearly independent columns.*

Proof (⇒)

- assume A_B has linearly dependent columns
- there exists $d \neq 0$ such that $A_B d = 0$
- extend d to $\mathbb{R}^n$ by adding 0-components
- now, $Ad = 0$ and $d_j = 0$ whenever $x_j = 0$
- for sufficiently small λ we have $x \pm \lambda d \in P$
- hence, x is not a vertex

Observation

For an LP we can assume wlog. that the matrix A has full row-rank. This means $\operatorname{rank}(A) = m$.

- assume that $\operatorname{rank}(A) < m$
- assume wlog. that the first row A_1 lies in the span of the other rows $A_2, \ldots, A_m$; this means

$$A_1 = \sum_{i=2}^{m} \lambda_i \cdot A_i\text{, for suitable } \lambda_i$$

C1 if now $b_1 = \sum_{i=2}^{m} \lambda_i \cdot b_i$ then for all x with $A_i x = b_i$ we also have $A_1 x = b_1$; hence the first constraint is superfluous

C2 if $b_1 \neq \sum_{i=2}^{m} \lambda_i \cdot b_i$ then the LP is infeasible, since for all x that fulfill constraints $A_2, \ldots, A_m$ we have

$$A_1 x = \sum_{i=2}^{m} \lambda_i \cdot A_i x = \sum_{i=2}^{m} \lambda_i \cdot b_i \neq b_1$$

From now on we will always assume that the constraint matrix of a standard form LP has full row rank.

Theorem 10

Given $P = \{x \mid Ax = b, x \geq 0\}$. x *is a vertex iff there exists* $B \subseteq \{1, \ldots, n\}$ *with* $|B| = m$ *and*

- A_B is non-singular
- $x_B = A_B^{-1}b \geq 0$
- $x_N = 0$

where $N = \{1, \ldots, n\} \setminus B$.

Proof

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\operatorname{rank}(A) = m$.

Basic Feasible Solutions

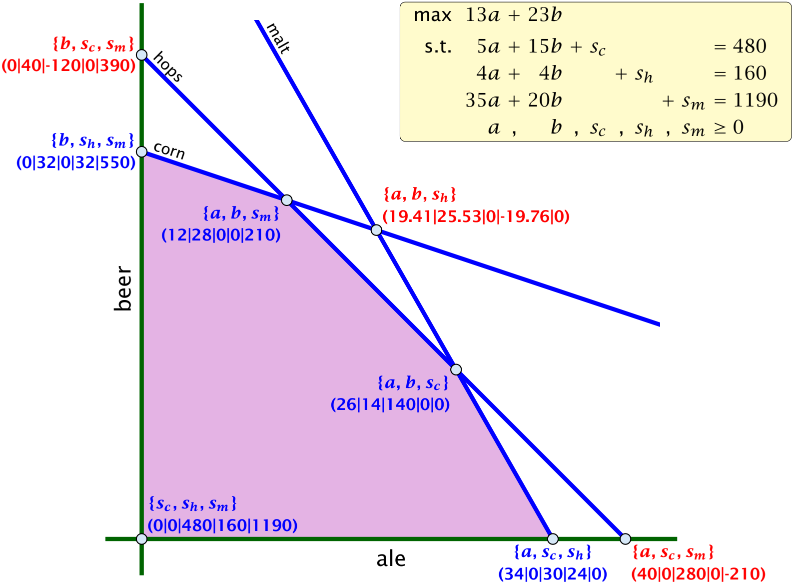
$x \in \mathbb{R}^n$ is called basic solution (Basislösung) if $Ax = b$ and $\operatorname{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a basic **feasible** solution (gültige Basislösung) if in addition $x \geq 0$.

A basis (Basis) is an index set $B \subseteq \{1, \ldots, n\}$ with $\operatorname{rank}(A_B) = m$ and $|B| = m$.

$x \in \mathbb{R}^n$ with $A_B x = b$ and $x_j = 0$ for all $j \notin B$ is the basic solution associated to basis B (die zu B assoziierte Basislösung)

Algebraic View



$$
\begin{array}{rrcrcrcrcrcl}
\text{max} & 13a & + & 23b & & & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
 & 4a & + & 4b & & & + & s_h & & & = & 160 \\
 & 35a & + & 20b & & & & & + & s_m & = & 1190 \\
 & a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0
\end{array}
$$

Fundamental Questions

Linear Programming Problem (LP)

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^t x \geq \alpha$?

Questions:

- Is LP in NP? yes!
- Is LP in co-NP?
- Is LP in P?

Proof:

- Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

Observation

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \text{poly}(n,m)\right)$.

- there are only $\binom{n}{m}$ different bases.
- compute the profit of each of them and take the maximum

4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

4 Simplex Algorithm

$$
\begin{array}{ll}
\text{max} & 13a + 23b \\
\text{s.t.} & 5a + 15b + s_c = 480 \\
 & 4a + 4b + s_h = 160 \\
 & 35a + 20b + s_m = 1190 \\
 & a, b, s_c, s_h, s_m \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{max } Z \\
13a + 23b - Z = 0 \\
5a + 15b + s_c = 480 \\
4a + 4b + s_h = 160 \\
35a + 20b + s_m = 1190 \\
a, b, s_c, s_h, s_m \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$A = B = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Pivoting Step

max Z

$$\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

max Z

$$\begin{array}{rcrcccccl}
13a & + & 23\boldsymbol{b} & & & & & -\; Z & = 0 \\
5a & + & 15\boldsymbol{b} & + s_c & & & & & = 480 \\
4a & + & 4\boldsymbol{b} & & + s_h & & & & = 160 \\
35a & + & 20\boldsymbol{b} & & & + s_m & & & = 1190 \\
a & , & \boldsymbol{b} & , s_c & , s_h & , s_m & & & \geq 0
\end{array}$$

basis = $\{s_c, s_h, s_m\}$

$a \;= b = 0$

$Z \;= 0$

$s_c \;= 480$

$s_h \;= 160$

$s_m = 1190$

- Choose variable with coefficient ≥ 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{{}^{480}/_{15}, {}^{160}/_{4}, {}^{1190}/_{20}\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{{}^{480}/_{15}, {}^{160}/_{4}, {}^{1190}/_{20}\}$ becomes the leaving variable.

max Z

$$\begin{array}{rcrcccccl}
13a & + & 23b & & & & & -\;Z & = 0 \\
5a & + & 15\boldsymbol{b} & + s_c & & & & & = 480 \\
4a & + & 4b & & + s_h & & & & = 160 \\
35a & + & 20b & & & + s_m & & & = 1190 \\
a & , & b & , s_c & , s_h & , s_m & & & \geq 0
\end{array}$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

max Z

$$\begin{array}{rcccccccl}
\frac{16}{3}a & & & - \frac{23}{15}s_c & & & -\;Z & = & -736 \\
\frac{1}{3}a & + & b & + \frac{1}{15}s_c & & & & = & 32 \\
\frac{8}{3}a & & & - \frac{4}{15}s_c & + s_h & & & = & 32 \\
\frac{85}{3}a & & & - \frac{4}{3}s_c & & + s_m & & = & 550 \\
a & , & b & , s_c & , s_h & , s_m & & \geq & 0
\end{array}$$

basis = $\{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

max Z

$$\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}\boldsymbol{a} & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}\boldsymbol{a} & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\boldsymbol{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}\boldsymbol{a} & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
\boldsymbol{a} & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis = $\{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

max Z

$$\begin{array}{rcrcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & - & Z & = & -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & & = & 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & & = & 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & & = & 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis = $\{a, b, s_m\}$

$s_c = s_h = 0$

$Z = 800$

$b = 28$

$a = 12$

$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^t x_B & + & c_N^t x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

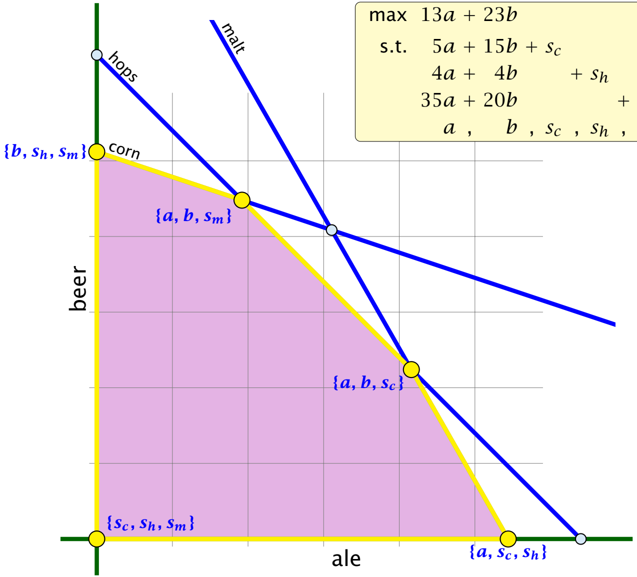
$$\begin{array}{rcrcl} & & (c_N^t - c_B^t A_B^{-1} A_N) x_N & = & Z - c_B^t A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^t - c_B^t A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting

$$
\begin{array}{llcl}
\text{max} & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & 4a + 4b + s_h & = & 160 \\
 & 35a + 20b + s_m & = & 1190 \\
 & a \;,\; b \;,\; s_c \;,\; s_h \;,\; s_m & \geq & 0
\end{array}
$$



Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
  - Other non-basis variables should stay at 0.
  - Basis variables change to maintain feasibility.
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for *d*:

- $d_j = 1$ (normalization)
- $d_\ell = 0$, $\ell \notin B$, $\ell \neq j$
- $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

Algebraic Definition of Pivoting

Definition 11 (j-th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1} A_{*j}$ is called the j-th basis direction for B.

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^t d = \theta(c_j - c_B^t A_B^{-1} A_{*j})$$

Algebraic Definition of Pivoting

Definition 12 (Reduced Cost)

For a basis B the value

$$\tilde{c}_j = c_j - c_B^t A_B^{-1} A_{*j}$$

is called the reduced cost for variable x_j.

Note that this is defined for every j. If $j \in B$ then the above term is 0.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{array}{rcrcl} c_B^t x_B & + & c_N^t x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^t - c_B^t A_B^{-1} A_N) x_N & = & Z - c_B^t A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^t - c_B^t A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Questions:

- What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- How do we find the initial basic feasible solution?
- Is there always a basis B such that

  $$(c_N^t - c_B^t A_B^{-1} A_N) \leq 0 \quad ?$$

  Then we can terminate because we know that the solution is optimal.
- If yes how do we make sure that we reach such a basis?

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

Termination

The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

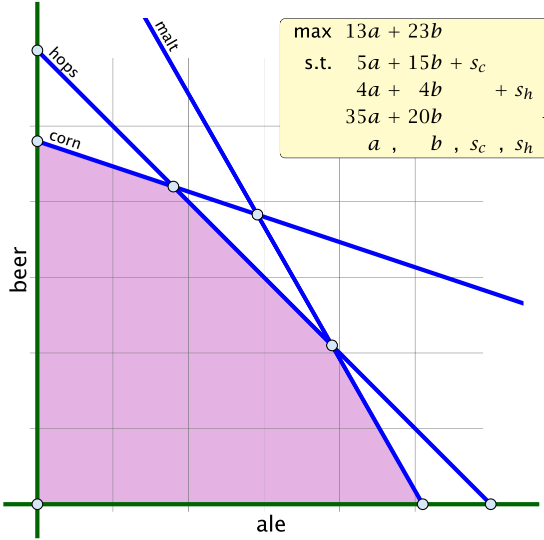
The set of inequalities is degenerate (also the basis is degenerate).

Definition 13 (Degeneracy)

A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

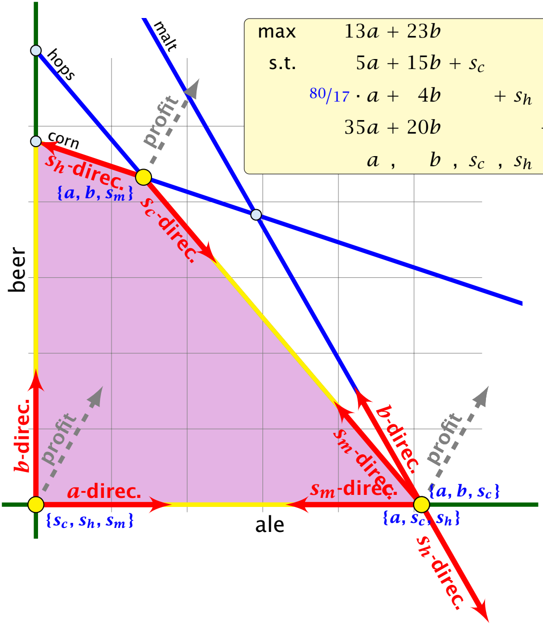
It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

Non Degenerate Example



$$\begin{array}{lrl} \text{max} & 13a + 23b & \\ \text{s.t.} & 5a + 15b + s_c & = 480 \\ & 4a + 4b + s_h & = 160 \\ & 35a + 20b + s_m & = 1190 \\ & a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq 0 \end{array}$$

Degenerate Example



$$
\begin{array}{llcl}
\text{max} & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & 80/17 \cdot a + 4b + s_h & = & 160 \\
 & 35a + 20b + s_m & = & 1190 \\
 & a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0
\end{array}
$$

Summary: How to choose pivot-elements

- We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
- The standard choice is the column that maximizes $\tilde{c}_e$.
- If $A_{ie} \leq 0$ for all $i \in \{1, \ldots, m\}$ then the maximum is not bounded.
- Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- If several variables have minimum $b_\ell / A_{\ell e}$ you reach a degenerate basis.
- Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Termination

What do we have so far?
Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

How do we come up with an initial solution?

- $Ax \leq b, x \geq 0$, and $\boldsymbol{b \geq 0}$.
- The standard slack from for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- Then $s = b$, $x = 0$ is a basic feasible solution (how?).
- We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

Two phase algorithm

Suppose we want to maximize $c^t x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1.
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b$, $x \geq 0$, $v \geq 0$ using Simplex. $x = 0$, $v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Optimality

Lemma 14

Let B be a basis and x^* a BFS corresponding to basis B. $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.

Duality

How do we get an upper bound to a maximization LP?

$$\begin{array}{rrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \leq 480 \\ & 4a & + & 4b & \leq 160 \\ & 35a & + & 20b & \leq 1190 \\ & & & a,b & \geq 0 \end{array}$$

Note that a lower bound is easy to derive. Every choice of $a, b \geq 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \geq 0$) such that $\sum_i y_i a_{ij} \geq c_j$ then $\sum_i y_i b_i$ will be an upper bound.

Duality

Definition 15
Let $z = \max\{c^t x \mid Ax \le b, x \ge 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^t y \mid A^t y \ge c, y \ge 0\}$$

is called the dual problem.

Duality

Lemma 16

The dual of the dual problem is the primal problem.

Proof:

- $w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$
- $w = -\max\{-b^t y \mid -A^t y \leq -c, y \geq 0\}$

The dual problem is

- $z = -\min\{-c^t x \mid -Ax \geq -b, x \geq 0\}$
- $z = \max\{c^t x \mid Ax \leq b, x \geq 0\}$

Weak Duality

Let $z = \max\{c^t x \mid Ax \le b, x \ge 0\}$ and
$w = \min\{b^t y \mid A^t y \ge c, y \ge 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \le b, x \ge 0\}$

y is dual feasible, iff $y \in \{y \mid A^t y \ge c, y \ge 0\}$.

Theorem 17 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^t \hat{x} \le z \le w \le b^t \hat{y} \ .$$

Weak Duality

$A^t\hat{y} \geq c \Rightarrow \hat{x}^tA^t\hat{y} \geq \hat{x}^tc$ $(\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^tA\hat{x} \leq \hat{y}^tb$ $(\hat{y} \geq 0)$

This gives

$$c^t\hat{x} \leq \hat{y}^tA\hat{x} \leq b^t\hat{y} \enspace .$$

Since, there exists primal feasible $\hat{x}$ with $c^t\hat{x} = z$, and dual feasible $\hat{y}$ with $b^ty = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

The following linear programs form a primal dual pair:

$$z = \max\{c^t x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^t y \mid A^t y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$
\begin{aligned}
&\max\{c^t x \mid Ax = b, x \geq 0\}\\
&\qquad = \max\{c^t x \mid Ax \leq b, -Ax \leq -b, x \geq 0\}\\
&\qquad = \max\{c^t x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}
\end{aligned}
$$

Dual:

$$
\begin{aligned}
&\min\{[b^t\ -b^t] y \mid [A^t\ -A^t] y \geq c, y \geq 0\}\\
&\qquad = \min\left\{[b^t\ -b^t] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^t\ -A^t] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\}\\
&\qquad = \min\left\{b^t \cdot (y^+ - y^-) \,\middle|\, A^t \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\}\\
&\qquad = \min\left\{b^t y' \,\middle|\, A^t y' \geq c\right\}
\end{aligned}
$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^t - c_B^t A_B^{-1} A \leq 0$$

This is equivalent to $A^t (A_B^{-1})^t c_B \geq c$

$y^* = (A_B^{-1})^t c_B$ is solution to the dual $\min\{b^t y | A^t y \geq c\}$.

$$\begin{aligned} b^t y^* &= (Ax^*)^t y^* = (A_B x_B^*)^t y^* \\ &= (A_B x_B^*)^t (A_B^{-1})^t c_B = (x_B^*)^t A_B^t (A_B^{-1})^t c_B \\ &= c^t x^* \end{aligned}$$

Hence, the solution is optimal.

Strong Duality

Theorem 18 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then

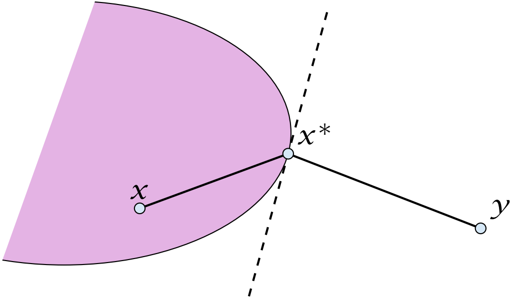
$$z^* = w^*$$

Lemma 19 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X. Then $\min\{f(x) : x \in X\}$ exists.

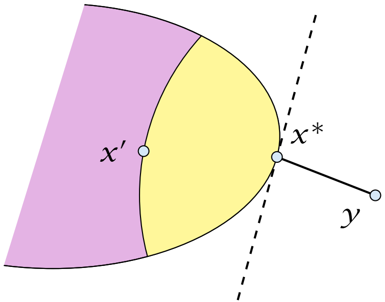
Lemma 20 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ *be a non-empty convex set, and let* $y \notin X$*. Then there exist* $x^* \in X$ *with minimum distance from* y*. Moreover for all* $x \in X$ *we have* $(y - x^*)^t(x - x^*) \leq 0$*.*



Proof of the Projection Lemma

- Define $f(x) = \|y - x\|$.
- We want to apply Weierstrass but X may not be bounded.
- $X \neq \emptyset$. Hence, there exists $x' \in X$.
- Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.
- Applying Weierstrass gives the existence.



Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2\|x - x^*\|^2 - 2\epsilon(y - x^*)^t(x - x^*)\end{aligned}$$

Hence, $(y - x^*)^t(x - x^*) \leq \frac{1}{2}\epsilon\|x - x^*\|^2$.

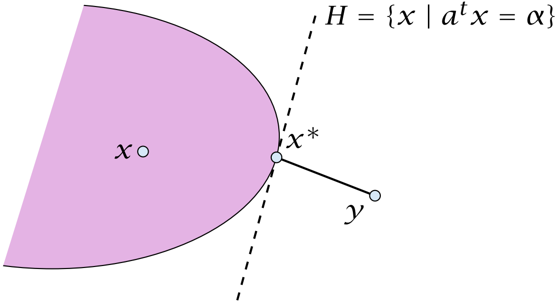
Letting $\epsilon \to 0$ gives the result.

Theorem 21 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^t x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X. ($a^t y < \alpha$; $a^t x \geq \alpha$ for all $x \in X$)

Proof of the Hyperplane Lemma

- Let $x^* \in X$ be closest point to y in X.
- By previous lemma $(y - x^*)^t(x - x^*) \leq 0$ for all $x \in X$.
- Choose $a = (x^* - y)$ and $\alpha = a^t x^*$.
- For $x \in X$: $a^t(x - x^*) \geq 0$, and, hence, $a^t x \geq \alpha$.
- Also, $a^t y = a^t(x^* - a) = \alpha - \|a\|^2 < \alpha$



Lemma 22 (Farkas Lemma)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ *with* $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ *with* $A^t y \geq 0$, $b^t y < 0$

Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^t b = y^t Ax \geq 0$$

Hence, at most one of the statements can hold.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

We want to show that there is y with $A^t y \geq 0$, $b^t y < 0$.

Let y be a hyperplane that separates b from S. Hence, $y^t b < \alpha$ and $y^t s \geq \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^t b < 0$

$y^t Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^t A \geq 0$ as we can choose x arbitrarily large.

Lemma 23 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ *with* $Ax \le b$, $x \ge 0$
2. $\exists y \in \mathbb{R}^m$ *with* $A^t y \ge 0$, $b^t y < 0$, $y \ge 0$

Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A \; I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \ge 0$, $s \ge 0$
2. $\exists y \in \mathbb{R}^m$ with $\begin{bmatrix} A^t \\ I \end{bmatrix} y \ge 0$, $b^t y < 0$

Proof of Strong Duality

$$P\colon\ z = \max\{c^t x \mid Ax \le b, x \ge 0\}$$

$$D\colon\ w = \min\{b^t y \mid A^t y \ge c, y \ge 0\}$$

Theorem 24 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D, respectively (i.e., P and D are non-empty). Then

$$z = w\ .$$

Proof of Strong Duality

$\boldsymbol{z \le w}$: follows from weak duality

$\boldsymbol{z \ge w}$:
We show $z < \alpha$ implies $w < \alpha$.

$$\begin{array}{lrcl} \exists x \in \mathbb{R}^n & & & \\ \text{s.t.} & Ax & \le & b \\ & -c^t x & \le & -\alpha \\ & x & \ge & 0 \end{array} \qquad \begin{array}{lrcl} \exists y \in \mathbb{R}^m; v \in \mathbb{R} & & & \\ \text{s.t.} & A^t y - cv & \ge & 0 \\ & b^t y - \alpha v & < & 0 \\ & y, v & \ge & 0 \end{array}$$

From the definition of α we know that the first system is infeasible; hence the second must be feasible.

Proof of Strong Duality

$$\begin{array}{rrcl} \exists y \in \mathbb{R}^m; v \in \mathbb{R} & & & \\ \text{s.t.} & A^t y - v & \geq & 0 \\ & b^t y - \alpha v & < & 0 \\ & y, v & \geq & 0 \end{array}$$

If the solution y, v has $v = 0$ we have that

$$\begin{array}{rrcl} \exists y \in \mathbb{R}^m & & & \\ \text{s.t.} & A^t y & \geq & 0 \\ & b^t y & < & 0 \\ & y & \geq & 0 \end{array}$$

is feasible. By Farkas lemma this gives that LP P is infeasible. Contradiction to the assumption of the lemma.

Proof of Strong Duality

Hence, there exists a solution y, v with $v > 0$.

We can rescale this solution (scaling both y and v) s.t. $v = 1$.

Then y is feasible for the dual but $b^t y < \alpha$. This means that $w < \alpha$.

Fundamental Questions

Definition 25 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^t x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP? yes!
- Is LP in P?

Proof:

- Given a primal maximization problem P and a parameter α. Suppose that $\alpha > \mathrm{opt}(P)$.
- We can prove this by providing an optimal basis for the dual.
- A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost $< \alpha$.

Complementary Slackness

Lemma 26

Assume a linear program $P = \max\{c^t x \mid Ax \le b; x \ge 0\}$ has solution x^* and its dual $D = \min\{b^t y \mid A^t y \ge c; y \ge 0\}$ has solution y^*.

1. *If $x_j^* > 0$ then the j-th constraint in D is tight.*
2. *If the j-th constraint in D is not tight than $x_j^* = 0$.*
3. *If $y_i^* > 0$ then the i-th constraint in P is tight.*
4. *If the i-th constraint in P is not tight than $y_i^* = 0$.*

If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a contraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^t x^* \le y^{*t} A x^* \le b^t y^*$$

Because of strong duality we then get

$$c^t x^* = y^{*t} A x^* = b^t y^*$$

This gives e.g.

$$\sum_j (y^t A - c^t)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^t A \ge c^t$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^t A - c^t)_j > 0$ (the j-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$\begin{array}{rrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \leq 480 \\ & 4a & + & 4b & \leq 160 \\ & 35a & + & 20b & \leq 1190 \\ & & & a,b & \geq 0 \end{array}$$

- Entrepeneur: buy resources from brewer at minimum cost
C, H, M: unit price for corn, hops and malt.

$$\begin{array}{rrcrcrl} \text{min} & 480C & + & 160H & + & 1190M & \\ \text{s.t.} & 5C & + & 4H & + & 35M & \geq 13 \\ & 15C & + & 4H & + & 20M & \geq 23 \\ & & & & & C,H,M & \geq 0 \end{array}$$

Note that brewer won't sell (at least not all) if e.g.
$5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

Interpretation of Dual Variables

Marginal Price:

- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by $\varepsilon_C, \varepsilon_H$, and ε_M, respectively.

The profit increases to $\max\{c^t x \mid Ax \le b + \varepsilon; x \ge 0\}$. Because of strong duality this is equal to

$$\begin{array}{lrcl} \min & (b^t + \epsilon^t)y & & \\ \text{s.t.} & A^t y & \ge & c \\ & y & \ge & 0 \end{array}$$

Interpretation of Dual Variables

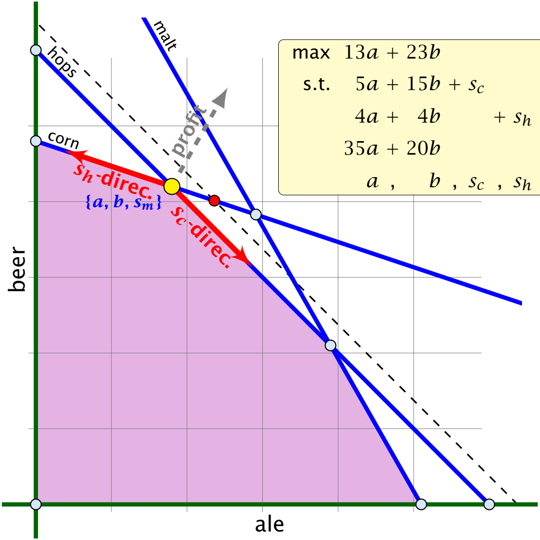
If ϵ is "small" enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

- If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
- If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the brewer. Hence, it makes no sense to have left-overs of this resource. Therefore its slack must be zero.

Example



$$
\begin{array}{llllllll}
\text{max} & 13a & + 23b & & & & & \\
\text{s.t.} & 5a & + 15b & + s_c & & & = & 480 \\
 & 4a & + \ \ 4b & & + s_h & & = & 160 \\
 & 35a & + 20b & & & + s_m & = & 1190 \\
 & a \ , & b \ , & s_c \ , & s_h \ , & s_m & \geq & 0
\end{array}
$$

The change in profit when increasing hops by one unit is

$$= \underbrace{c_B^t A_B^{-1}}_{y^*} e_h .$$

Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

Flows

Definition 27

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)

$$0 \leq f_{xy} \leq c_{xy} \enspace .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} \enspace .$$

   (flow conservation constraints)

Flows

Definition 28

The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \quad .$$

Maximum Flow Problem:

Find an (s,t)-flow with maximum value.

LP-Formulation of Maxflow

$$\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
 & \forall w \neq s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
 & & f_{zw} & \ge & 0 &
\end{array}$$

$$\begin{array}{lllcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} \qquad\quad +1p_y & \ge & 1 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} \qquad\quad +1p_y & \ge & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
 & f_{st}: & 1\ell_{st} & \ge & 1 \\
 & f_{ts}: & 1\ell_{ts} & \ge & -1 \\
 & & \ell_{xy} & \ge & 0
\end{array}$$

LP-Formulation of Maxflow

$$
\begin{array}{lllrcl}
\min & & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy} \ (x,y \neq s,t): & & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy} \ (y \neq s,t): & & 1\ell_{sy} - \ \ 1 + 1p_y & \geq & 0 \\
 & f_{xs} \ (x \neq s,t): & & 1\ell_{xs} - 1p_x + \ \ 1 & \geq & 0 \\
 & f_{ty} \ (y \neq s,t): & & 1\ell_{ty} - \ \ 0 + 1p_y & \geq & 0 \\
 & f_{xt} \ (x \neq s,t): & & 1\ell_{xt} - 1p_x + \ \ 0 & \geq & 0 \\
 & f_{st}: & & 1\ell_{st} - \ \ 1 + \ \ 0 & \geq & 0 \\
 & f_{ts}: & & 1\ell_{ts} - \ \ 0 + \ \ 1 & \geq & 0 \\
 & & & \ell_{xy} & \geq & 0
\end{array}
$$

LP-Formulation of Maxflow

$$\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - p_s + 1p_y & \geq & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + p_s & \geq & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - p_t + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + p_t & \geq & 0 \\
 & f_{st}: & 1\ell_{st} - p_s + p_t & \geq & 0 \\
 & f_{ts}: & 1\ell_{ts} - p_t + p_s & \geq & 0 \\
 & & \ell_{xy} & \geq & 0
\end{array}$$

with $p_t = 0$ and $p_s = 1$.

LP-Formulation of Maxflow

$$\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & \textcolor{red}{f_{xy}:} & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& & \ell_{xy} & \geq & 0 \\
& & p_s & = & 1 \\
& & p_t & = & 0
\end{array}$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.

Degeneracy Revisited

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

Idea:

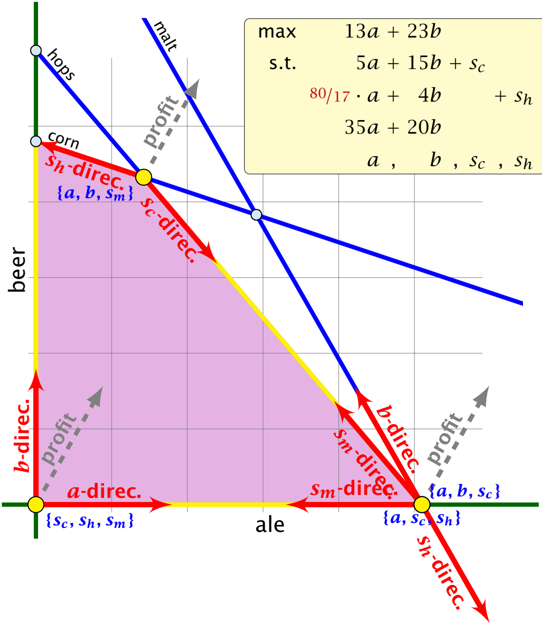
Change $\mathrm{LP} := \max\{c^t x, Ax = b; x \geq 0\}$ into $\mathrm{LP}' := \max\{c^t x, Ax = b', x \geq 0\}$ such that

I. LP is feasible

II. If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1} b \not\geq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).

III. LP has no degenerate basic solutions

Degenerate Example



$$
\begin{array}{llcl}
\text{max} & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & 80/17 \cdot a + 4b + s_h & = & 160 \\
 & 35a + 20b + s_m & = & 1190 \\
 & a \,,\; b \,,\; s_c \,,\; s_h \,,\; s_m & \geq & 0
\end{array}
$$

Degeneracy Revisited

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

Idea:
Given feasible $\mathrm{LP} := \max\{c^t x, Ax = b; x \geq 0\}$. Change it into $\mathrm{LP}' := \max\{c^t x, Ax = b', x \geq 0\}$ such that

- **I.** LP' is feasible
- **II.** If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1} b \ngeq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).
- **III.** LP' has no degenerate basic solutions

Perturbation

Let B be index set of some basis with basic solution

$$x_B^* = A_B^{-1}b \geq 0, x_N^* = 0 \quad \text{(i.e. } B \text{ is feasible)}$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \text{ for } \varepsilon > 0 \ .$$

This is the perturbation that we are using.

Property I

The new LP is feasible because the set B of basis variables provides a feasible basis:

$$A_B^{-1}\left(b + A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right) = x_B^* + \begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix} \geq 0 \enspace .$$

Property II

Let $\tilde{B}$ be a non-feasible basis. This means $(A_{\tilde{B}}^{-1}b)_i < 0$ for some row i.

Then for small enough $\epsilon > 0$

$$\left(A_{\tilde{B}}^{-1}\left(b + A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)\right)_i = (A_{\tilde{B}}^{-1}b)_i + \left(A_{\tilde{B}}^{-1}A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)_i < 0$$

Hence, $\tilde{B}$ is not feasible.

Property III

Let $\tilde{B}$ be a basis. It has an associated solution

$$x^*_{\tilde{B}} = A^{-1}_{\tilde{B}} b + A^{-1}_{\tilde{B}} A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynom with variable ε of degree at most m.

$A^{-1}_{\tilde{B}} A_B$ has rank m. Therefore no polynom is 0.

A polynom of degree at most m has at most m roots (Nullstellen).

Hence, $\epsilon > 0$ small enough gives that no component of the above vector is 0. Hence, no degeneracies.

Since, there are no degeneracies Simplex will terminate when run on LP'.

- If it terminates because the reduced cost vector fulfills

  $$\tilde{c} = (c^t - c_B^t A_B^{-1} A) \leq 0$$

  then we have found an optimal basis. Note that this basis is also optimal for LP, as the above constraint does not depend on b.
- If it terminates because it finds a variable x_j with $\tilde{c}_j > 0$ for which the j-th basis direction d, fulfills $d \geq 0$ we know that LP' is unbounded. The basis direction does not depend on b. Hence, we also know that LP is unbounded.

Lexicographic Pivoting

Doing calculations with perturbed instances may be costly. Also the right choice of ε is difficult.

Idea:
Simulate behaviour of LP' without explicitly doing a perturbation.

Lexicographic Pivoting

We choose the entering variable arbitrarily as before ($\tilde{c}_e > 0$, of course).

If we do not have a choice for the leaving variable then LP' and LP do the same (i.e., choose the same variable).

Otherwise we have to be careful.

Lexicographic Pivoting

In the following we assume that $b \geq 0$. This can be obtained by replacing the initial system $(A_B \mid b)$ by $(A_B^{-1}A \mid A_B^{-1}b)$ where B is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

Then the perturbed instance is

$$b' = b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^t x_B & + & c_N^t x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^t - c_B^t A_B^{-1} A_N) x_N & = & Z - c_B^t A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^t - c_B^t A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Lexicographic Pivoting

LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

$$\theta_\ell = \frac{\hat{b}_\ell}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_\ell}{(A_B^{-1}A_{*e})_\ell} \enspace .$$

ℓ is the index of a leaving variable within B. This means if e.g. $B = \{1, 3, 7, 14\}$ and leaving variable is 3 then $\ell = 2$.

Lexicographic Pivoting

Definition 29

$u \leq_{\text{lex}} v$ if and only if the first component in which u and v differ fulfills $u_i \leq v_i$.

Lexicographic Pivoting

LP' chooses an index that minimizes

$$\theta_\ell = \frac{\left(A_B^{-1}\left(b + \begin{pmatrix}\varepsilon \\ \vdots \\ \varepsilon^m\end{pmatrix}\right)\right)_\ell}{(A_B^{-1}A_{*e})_\ell} = \frac{\left(A_B^{-1}(b \mid I)\begin{pmatrix}1 \\ \varepsilon \\ \vdots \\ \varepsilon^m\end{pmatrix}\right)_\ell}{(A_B^{-1}A_{*e})_\ell}$$

$$= \frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}\begin{pmatrix}1 \\ \varepsilon \\ \vdots \\ \varepsilon^m\end{pmatrix}$$

Lexicographic Pivoting

This means you can choose the variable/row ℓ for which the vector

$$\frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}$$

is lexicographically minimal.

Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

Number of Simplex Iterations

Each iteration of Simplex can be implemented in polynomial time.

If we use lexicographic pivoting we know that Simplex requires at most $\binom{n}{m}$ iterations, because it will not visit a basis twice.

The input size is $L \cdot n \cdot m$, where n is the number of variables, m is the number of constraints, and L is the length of the binary representation of the largest coefficient in the matrix A.

If we really require $\binom{n}{m}$ iterations then Simplex is not a polynomial time algorithm.

Can we obtain a better analysis?

Number of Simplex Iterations

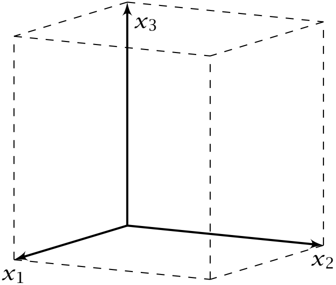
Observation

Simplex visits every feasible basis at most once.

However, also the number of feasible bases can be very large.

Example

$$\begin{aligned} \max\ & c^t x \\ \text{s.t.}\ & 0 \le x_1 \le 1 \\ & 0 \le x_2 \le 1 \\ & \vdots \\ & 0 \le x_n \le 1 \end{aligned}$$

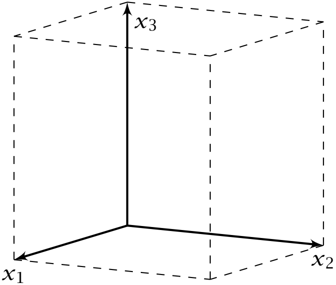


$2n$ constraint on n variables define an n-dimensional hypercube as feasible region.

The feasible region has 2^n vertices.

Example

$$\begin{aligned} \max\ & c^t x \\ \text{s.t.}\ & 0 \le x_1 \le 1 \\ & 0 \le x_2 \le 1 \\ & \vdots \\ & 0 \le x_n \le 1 \end{aligned}$$



However, Simplex may still run quickly as it usually does not visit all feasible bases.

In the following we give an example of a feasible region for which there is a bad Pivoting Rule.

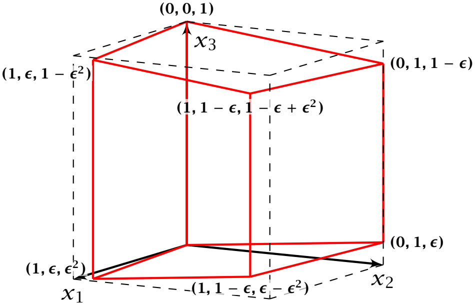
Pivoting Rule

A Pivoting Rule defines how to choose the entering and leaving variable for an iteration of Simplex.

In the non-degenerate case after choosing the entering variable the leaving variable is unique.

Klee Minty Cube

$$\begin{array}{lrcl}
\text{max } x_n & & & \\
\quad \text{s.t.} & 0 \le & x_1 & \le 1 \\
& \epsilon x_1 \le & x_2 & \le 1 - \epsilon x_1 \\
& \epsilon x_2 \le & x_3 & \le 1 - \epsilon x_2 \\
& & \vdots & \\
& \epsilon x_{n-1} \le & x_n & \le 1 - \epsilon x_{n-1} \\
& & x_i & \ge 0
\end{array}$$



Observations

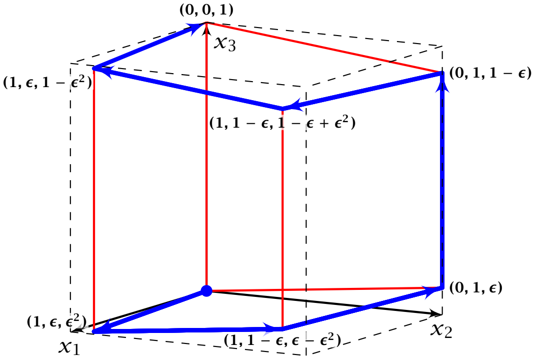
- We have $2n$ constraints, and $3n$ variables (after adding slack variables to every constraint).
- Every basis is defined by $2n$ variables, and n non-basic variables.
- There exist degenerate vertices.
- The degeneracies come from the non-negativity constraints, which are superfluous.
- In the following all variables x_i stay in the basis at all times.
- Then, we can uniquely specify a basis by choosing for each variable whether it should be equal to its lower bound, or equal to its upper bound (the slack variable corresponding to the non-tight constraint is part of the basis).
- We can also simply identify each basis/vertex with the corresponding hypercube vertex obtained by letting $\epsilon \to 0$.

Analysis

- In the following we specify a sequence of bases (identified by the corresponding hypercube node) along which the objective function strictly increases.
- The basis $(0,\dots,0,1)$ is the unique optimal basis.
- Our sequence S_n starts at $(0,\dots,0)$ ends with $(0,\dots,0,1)$ and visits every node of the hypercube.
- An unfortunate Pivoting Rule may choose this sequence, and, hence, require an exponential number of iterations.

Klee Minty Cube

$$
\begin{array}{rrcl}
\text{max } x_n & & & \\
\text{s.t.} & 0 \le & x_1 & \le 1 \\
& \epsilon x_1 \le & x_2 & \le 1 - \epsilon x_1 \\
& \epsilon x_2 \le & x_3 & \le 1 - \epsilon x_2
\end{array}
$$



Analysis

The sequence S_n that visits every node of the hypercube is defined recursively

$$\begin{tikzcd}
(0,\dots,0,0,0) \arrow[d, squiggly, "S_{n-1}"] \arrow[ddd, squiggly, bend left=60, "S_n"] \\
(0,\dots,0,1,0) \arrow[d] \\
(0,\dots,0,1,1) \arrow[d, squiggly, "S_{n-1}^{\text{rev}}"] \\
(0,\dots,0,0,1)
\end{tikzcd}$$

The non-recursive case is $S_1 = 0 \to 1$

Analysis

Lemma 30

The objective value x_n is increasing along path S_n.

Proof by induction:

$\boldsymbol{n = 1}$: obvious, since $S_1 = 0 \to 1$, and $1 > 0$.

$\boldsymbol{n - 1 \to n}$

- For the first part the value of $x_n = \epsilon x_{n-1}$.
- By induction hypothesis x_{n-1} is increasing along S_{n-1}, hence, also x_n.
- Going from $(0, \dots, 0, 1, 0)$ to $(0, \dots, 0, 1, 1)$ increases x_n for small enough ϵ.
- For the remaining path S_{n-1}^{rev} we have $x_n = 1 - \epsilon x_{n-1}$.
- By induction hypothesis x_{n-1} is increasing along S_{n-1}, hence $-\epsilon x_{n-1}$ is increasing along S_{n-1}^{rev}.

Remarks about Simplex

Observation
The simplex algorithm takes at most $\binom{n}{m}$ iterations. Each iteration can be implemented in time $\mathcal{O}(mn)$.

In practise it usually takes a linear number of iterations.

Remarks about Simplex

Theorem

For almost all known deterministic pivoting rules (rules for choosing entering and leaving variables) there exist lower bounds that require the algorithm to have exponential running time ($\Omega(2^{\Omega(n)})$) (e.g. Klee Minty 1972).

Remarks about Simplex

Theorem
For some standard randomized pivoting rules there exist subexponential lower bounds ($\Omega(2^{\Omega(n^{\alpha})})$ for $\alpha > 0$) (Friedmann, Hansen, Zwick 2011).

Remarks about Simplex

Conjecture (Hirsch 1957)
The edge-vertex graph of an m-facet polytope in d-dimensional Euclidean space has diameter no more than $m - d$.

The conjecture has been proven wrong in 2010.

But the question whether the diameter is perhaps of the form $\mathcal{O}(\mathrm{poly}(m, d))$ is open.

8 Seidels LP-algorithm

- Suppose we want to solve $\min\{c^t x \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have m constraints.
- In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
- If d is much smaller than m one can do a lot better.
- In the following we develop an algorithm with running time $\mathcal{O}(d! \cdot m)$, i.e., linear in m.

8 Seidels LP-algorithm

Setting:

- We assume an LP of the form

$$\boxed{\begin{array}{lrcl} \min & c^t x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}}$$

- We assume that the LP is bounded.

Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{lrcl} \min & c^t x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^t x$ for any basic feasible solution.**

Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b.

Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

Add slack variables to A; denote the resulting matrix with $\bar{A}$.

If B is an optimal basis then x_B with $\bar{A}_B x_B = b$, gives an optimal assignment to the basis variables (non-basic variables are 0).

Theorem 31 (Cramers Rule)

Let M be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by

$$x_j = \frac{\det(M_j)}{\det(M)} \enspace ,$$

where M_j is the matrix obtained from M by replacing the j-th column by the vector b.

Proof:

- Define
$$X_j = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{j-1} & x & e_{j+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$
Note that expanding along the j-th column gives that $\det(X_j) = x_j$.
- Further, we have
$$MX_j = \begin{pmatrix} | & & | & | & | & & | \\ Me_1 & \cdots & Me_{j-1} & Mx & Me_{j+1} & \cdots & Me_n \\ | & & | & | & | & & | \end{pmatrix} = M_j$$
- Hence,
$$x_j = \det(X_j) = \frac{\det(M_j)}{\det(M)}$$

Bounding the Determinant

Let Z be the maximum absolute entry occuring in $\bar{A}, \bar{b}$ or c. Let C denote the matrix obtained from $\bar{A}_B$ by replacing the j-th column with vector $\bar{b}$.

Observe that

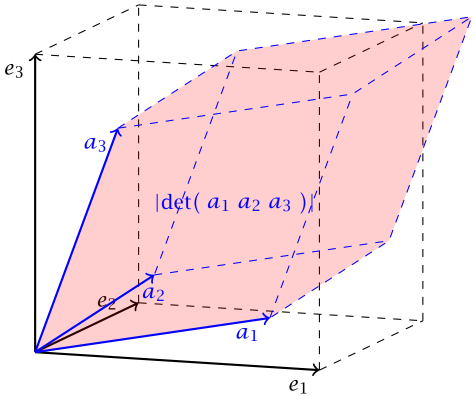
$$\begin{aligned} |\det(C)| &= \left| \sum_{\pi \in S_m} \prod_{1 \le i \le m} \operatorname{sgn}(\pi) C_{i\pi(i)} \right| \\ &\le \sum_{\pi \in S_m} \prod_{1 \le i \le m} |C_{i\pi(i)}| \\ &\le m! \cdot Z^m \quad . \end{aligned}$$

Bounding the Determinant

Alternatively, Hadamards inequality gives

$$\begin{aligned}|\det(C)| &\leq \prod_{i=1}^{m} \|C_{*i}\| \leq \prod_{i=1}^{m} (\sqrt{m}Z) \\ &\leq m^{m/2} Z^m \quad .\end{aligned}$$

Hadamards Inequality



Hadamards inequality says that the volume of the red parallelepiped (Spat) is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{rrcl} \min & c^t x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^t x$ for any basic feasible solution.** Add the constraint $c^t x \geq -mZ(m! \cdot Z^m) - 1$. Note that this constraint is superfluous unless the LP is unbounded.

Ensuring Conditions

Compute an optimum basis for the new LP.

- If the cost is $c^tx = -(mZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
- Otw. we have an optimum basis.

In the following we use $\mathcal{H}$ to denote the set of all constraints apart from the constraint $c^t x \geq -mZ(m! \cdot Z^m) - 1$.

We give a routine SeidelLP$(\mathcal{H}, d)$ that is given a set $\mathcal{H}$ of explicit, non-degenerate constraints over d variables, and minimizes $c^t x$ over all feasible points.

In addition it obeys the implicit constraint $c^t x \geq -(mZ)(m! \cdot Z^m) - 1$.

Algorithm 1 SeidelLP($\mathcal{H}, d$)

```
if $d = 1$ then solve 1-dimensional problem and return;
if $\mathcal{H} = \emptyset$ then return $x$ on implicit constraint hyperplane
choose random constraint $h \in \mathcal{H}$
$\hat{\mathcal{H}} \leftarrow \mathcal{H} \setminus \{h\}$
$\hat{x}^* \leftarrow$ SeidelLP($\hat{\mathcal{H}}, d$)
if $\hat{x}^* =$ infeasible then return infeasible
if $\hat{x}^*$ fulfills $h$ then return $\hat{x}^*$
// optimal solution fulfills $h$ with equality, i.e., $A_h x = b_h$
solve $A_h x = b_h$ for some variable $x_\ell$;
eliminate $x_\ell$ in constraints from $\hat{\mathcal{H}}$ and in implicit constr.;
$\hat{x}^* \leftarrow$ SeidelLP($\hat{\mathcal{H}}, d - 1$)
if $\hat{x}^* =$ infeasible then
    return infeasible
else
    add the value of $x_\ell$ to $\hat{x}^*$ and return the solution
```

8 Seidels LP-algorithm

- If $d = 1$ we can solve the 1-dimensional problem in time $\mathcal{O}(m)$.
- If $d > 1$ and $m = 0$ we take time $\mathcal{O}(d)$ to return d-dimensional vector x.
- The first recursive call takes time $T(m-1, d)$ for the call plus $\mathcal{O}(d)$ for checking whether the solution fulfills h.
- If we are unlucky and $\hat{x}^*$ does not fulfill h we need time $\mathcal{O}(d(m+1)) = \mathcal{O}(dm)$ to eliminate x_ℓ. Then we make a recursive call that takes time $T(m-1, d-1)$.
- The probability of being unlucky is at most d/m as there are at most d constraints whose removal will decrease the objective function

8 Seidels LP-algorithm

This gives the recurrence

$$T(m,d) = \begin{cases} \mathcal{O}(m) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m-1,d) + & \\ \frac{d}{m}(\mathcal{O}(dm) + T(m-1,d-1)) & \text{otw.} \end{cases}$$

Note that $T(m,d)$ denotes the expected running time.

8 Seidels LP-algorithm

Let C be the largest constant in the $\mathcal{O}$-notations.

$$T(m,d) = \begin{cases} Cm & \text{if } d = 1 \\ Cd & \text{if } d > 1 \text{ and } m = 0 \\ Cd + T(m-1,d) + & \\ \frac{d}{m}(Cdm + T(m-1,d-1)) & \text{otw.} \end{cases}$$

Note that $T(m,d)$ denotes the expected running time.

8 Seidels LP-algorithm

Let C be the largest constant in the $\mathcal{O}$-notations.

We show $T(m,d) \leq Cf(d)\max\{1,m\}$.

$d = 1$:

$$T(m,1) \leq Cm \leq Cf(1)\max\{1,m\} \text{ for } f(1) \geq 1$$

$d > 1; m = 0$:

$$T(0,d) \leq \mathcal{O}(d) \leq Cd \leq Cf(d)\max\{1,m\} \text{ for } f(d) \geq d$$

$d > 1; m = 1$:

$$\begin{aligned} T(1,d) &= \mathcal{O}(d) + T(0,d) + d\Big(\mathcal{O}(d) + T(0,d-1)\Big) \\ &\leq Cd + Cd + Cd^2 + dCf(d-1) \\ &\leq Cf(d)\max\{1,m\} \text{ for } f(d) \geq 3d^2 + df(d-1) \end{aligned}$$

8 Seidels LP-algorithm

$d > 1; m > 1$:
(by induction hypothesis statm. true for $d' < d, m' \geq 0$;
and for $d' = d$, $m' < m$)

$$\begin{aligned}
T(m,d) &= \mathcal{O}(d) + T(m-1,d) + \frac{d}{m}\Big(\mathcal{O}(dm) + T(m-1,d-1)\Big) \\
&\leq Cd + Cf(d)(m-1) + Cd^2 + \frac{d}{m}Cf(d-1)(m-1) \\
&\leq 2Cd^2 + Cf(d)(m-1) + dCf(d-1) \\
&\leq Cf(d)m
\end{aligned}$$

if $f(d) \geq df(d-1) + 2d^2$.

8 Seidels LP-algorithm

- Define $f(1) = 3 \cdot 1^2$ and $f(d) = df(d-1) + 3d^2$ for $d > 1$.

Then

$$\begin{aligned}
f(d) &= 3d^2 + df(d-1) \\
&= 3d^2 + d\Big[3(d-1)^2 + (d-1)f(d-2)\Big] \\
&= 3d^2 + d\Big[3(d-1)^2 + (d-1)\Big[3(d-2)^2 + (d-2)f(d-3)\Big]\Big] \\
&= 3d^2 + 3d(d-1)^2 + 3d(d-1)(d-2)^2 + \dots \\
&\quad + 3d(d-1)(d-2) \cdot \ldots \cdot 4 \cdot 3 \cdot 1^2 \\
&= 3d!\left(\frac{d^2}{d!} + \frac{(d-1)^2}{(d-1)!} + \frac{(d-2)^2}{(d-2)!} + \dots\right) \\
&= \mathcal{O}(d!)
\end{aligned}$$

since $\sum_{i \geq 1} \frac{i^2}{i!}$ is a constant.

Complexity

LP Feasibility Problem (LP feasibility)

- Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}$ with $Ax = b$, $x \geq 0$?
- Note that allowing A, b to contain rational numbers does not make a difference, as we can multiply every number by a suitable large constant so that everything becomes integral but the feasible region does not change.

Is this problem in NP or even in P?

The Bit Model

Input size

- The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- Let for an $m \times n$ matrix M, $L(M)$ denote the number of bits required to encode all the numbers in M.

$$L(M) := \sum_{i,j} \lceil \log_2(|m_{ij}|) + 1 \rceil$$

- In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
- Then the input length is $\Theta(L([A|b]))$.

- In the following we sometimes refer to $L := L([A|b])$ as the input size (even though the real input size is something in $\Theta(L([A|b]))$).
- In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in $L([A|b])$).

Suppose that $Ax = b$; $x \geq 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

$$x_B = A_B^{-1} b$$

and all other entries in x are 0.

Size of a Basic Feasible Solution

Lemma 32

Let $M \in \mathbb{Z}^{m\times m}$ be an invertable matrix and let $b \in \mathbb{Z}^m$. Further define $L' = L([M \mid b]) + n\log_2 n$. Then a solution to $Mx = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \le 2^{L'}$ and $|D| \le 2^{L'}$.

Proof:

Cramers rules says that we can compute x_j as

$$x_j = \frac{\det(M_j)}{\det(M)}$$

where M_j is the matrix obtained from M by replacing the j-th column by the vector b.

Bounding the Determinant

Let $X = A_B$. Then

$$\begin{aligned}
|\det(X)| &= \left|\sum_{\pi\in S_n}\prod_{1\le i\le n}\operatorname{sgn}(\pi)X_{i\pi(i)}\right| \\
&\le \sum_{\pi\in S_n}\prod_{1\le i\le n}|X_{i\pi(i)}| \\
&\le n!\cdot 2^{L([A|b])} \le n^n 2^L \le 2^{L'} \enspace .
\end{aligned}$$

Analogously for $\det(M_j)$.

This means if $Ax = b$, $x \geq 0$ is feasible we only need to consider vectors x where an entry x_j can be represented by a rational number with encoding length polynomial in the input length L.

Hence, the x that we have to guess is of length polynomial in the input-length L.

For a given vector x of polynomial length we can check for feasibility in polynomial time.

Hence, LP feasibility is in NP.

Reducing LP-solving to LP decision.

Given an LP $\max\{c^t x \mid Ax = b; x \geq 0\}$ do a binary search for the optimum solution

(Add constraint $c^t x - \delta = M$; $\delta \geq 0$ or ($c^t x \geq M$). Then checking for feasibility shows whether optimum solution is larger or smaller than M).

If the LP is feasible then the binary search finishes in at most

$$\log_2\left(\frac{2n2^{2L'}}{1/2^{L'}}\right) = \mathcal{O}(L') \ ,$$

as the range of the search is at most $-n2^{2L'}, \dots, n2^{2L'}$ and the distance between two adjacent values is at least $\frac{1}{\det(A)} \geq \frac{1}{2^{L'}}$.

Here we use $L' = L([A \mid b \mid c]) + n \log_2 n$ (it also includes the encoding size of c).

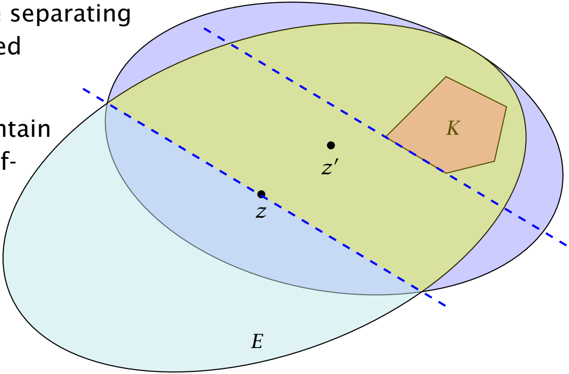
How do we detect whether the LP is unbounded?

Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

We can add a constraint $c^t x \geq M_{\max} + 1$ and check for feasibility.

Ellipsoid Method

- Let K be a convex set.
- Maintain ellipsoid E that is guaranteed to contain K provided that K is non-empty.
- If center $z \in K$ STOP.
- Otw. find a hyperplane separating K from z (e.g. a violated constraint in the LP).
- Shift hyperplane to contain node z. H denotes half-space that contains K.
- Compute (smallest) ellipsoid E' that contains $K \cap H$.
- REPEAT



Issues/Questions:

- How do you choose the first Ellipsoid? What is its volume?
- What if the polytop K is unbounded?
- How do you measure progress? By how much does the volume decrease in each iteration?
- When can you stop? What is the minimum volume of a non-empty polytop?

Definition 33

A mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(x) = Lx + t$, where L is an invertible matrix is called an affine transformation.

Definition 34

A ball in $\mathbb{R}^n$ with center c and radius r is given by

$$\begin{aligned} B(c,r) &= \{x \mid (x-c)^t(x-c) \le r^2\} \\ &= \{x \mid \sum_i (x-c)_i^2/r^2 \le 1\} \end{aligned}$$

$B(0,1)$ is called the unit ball.

Definition 35

An affine transformation of the unit ball is called an ellipsoid.

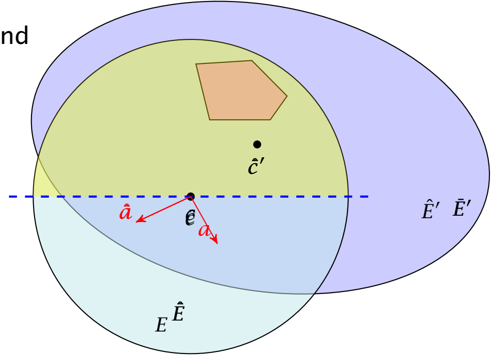
From $f(x) = Lx + t$ follows $x = L^{-1}(f(x) - t)$.

$$\begin{aligned} f(B(0,1)) &= \{f(x) \mid x \in B(0,1)\} \\ &= \{y \in \mathbb{R}^n \mid L^{-1}(y - t) \in B(0,1)\} \\ &= \{y \in \mathbb{R}^n \mid (y - t)^t L^{-1^t} L^{-1}(y - t) \le 1\} \\ &= \{y \in \mathbb{R}^n \mid (y - t)^t Q^{-1}(y - t) \le 1\} \end{aligned}$$

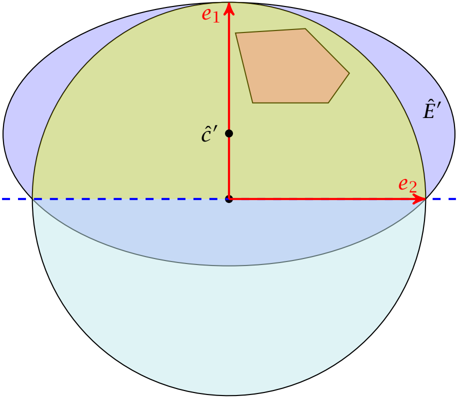
where $Q = LL^t$ is an invertible matrix.

How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
- Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1.
- Compute the new center $\hat{c}'$ and the new matrix $\hat{Q}'$ for this simplified setting.
- Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E.



The Easy Case



- The new center lies on axis x_1. Hence, $\hat{c}' = te_1$ for $t > 0$.
- The vectors $e_1, e_2, \dots$ have to fulfill the ellipsoid constraint with equality. Hence $(e_i - \hat{c}')^t \hat{Q}'^{-1}(e_i - \hat{c}') = 1$.

The Easy Case

- The obtain the matrix $\hat{Q}'^{-1}$ for our ellipsoid $\hat{E}'$ note that $\hat{E}'$ is axis-parallel.
- Let a denote the radius along the x_1-axis and let b denote the (common) radius for the other axes.
- The matrix
  $$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$
  maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

The Easy Case

- As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix $\hat{Q}'^{-1}$ is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

The Easy Case

- $(e_1 - \hat{c}')^t \hat{Q}'^{-1} (e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^t \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $(1-t)^2 = a^2$.

The Easy Case

- For $i \neq 1$ the equation $(e_i - \hat{c}')^t \hat{Q}'^{-1}(e_i - \hat{c}') = 1$ gives

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^t \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2} = 1 - \frac{t^2}{(1-t)^2} = \frac{1-2t}{(1-t)^2}$$

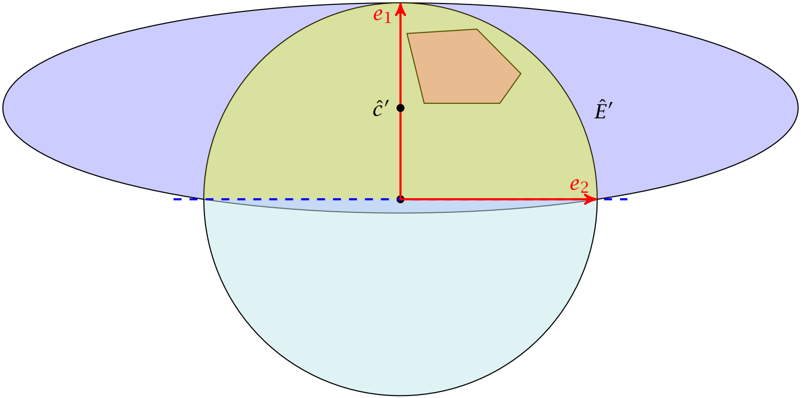
Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1-t}{\sqrt{1-2t}}$$

The Easy Case

We still have many choices for t:



Choose t such that the volume of $\hat{E}'$ is minimal!!!

The Easy Case

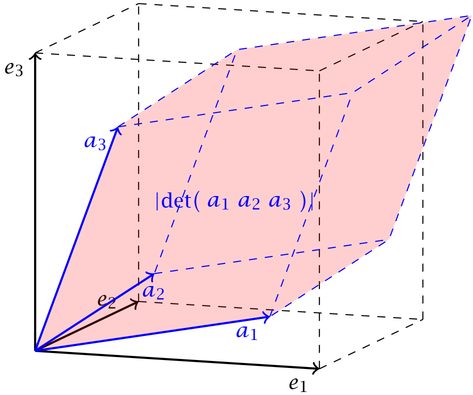
We want to choose t such that the volume of $\hat{E}'$ is minimal.

Lemma 36

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\mathrm{vol}(L(K)) = |\det(L)| \cdot \mathrm{vol}(K) \ .$$

n-dimensional volume



The Easy Case

- We want to choose t such that the volume of $\hat{E}'$ is minimal.

$$\mathrm{vol}(\hat{E}') = \mathrm{vol}(B(0,1)) \cdot |\det(\hat{L}')| \enspace ,$$

  where $\hat{Q}' = \hat{L}'\hat{L}'^t$.

- We have

$$\hat{L}'^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & \dots & 0 \\ 0 & \frac{1}{b} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b} \end{pmatrix} \text{ and } \hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

- Note that a and b in the above equations depend on t, by the previous equations.

The Easy Case

$$\begin{aligned}
\mathrm{vol}(\hat{E}') &= \mathrm{vol}(B(0,1)) \cdot |\det(\hat{L}')| \\
&= \mathrm{vol}(B(0,1)) \cdot ab^{n-1} \\
&= \mathrm{vol}(B(0,1)) \cdot (1-t) \cdot \left(\frac{1-t}{\sqrt{1-2t}}\right)^{n-1} \\
&= \mathrm{vol}(B(0,1)) \cdot \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}
\end{aligned}$$

The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

$$= \frac{1}{N^2}\cdot\left((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}}\right.$$

derivative of numerator

denominator

N = denominator

$$\left.\cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^n}}\right)$$

outer derivative

inner derivative

numerator

$$= \frac{1}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}$$

$$\cdot\Big((n-1)(1-t)-n(1-2t)\Big)$$

$$= \frac{1}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)$$

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a = 1 - t = \frac{n}{n+1} \text{ and } b = \frac{1-t}{\sqrt{1-2t}} = \frac{n}{\sqrt{n^2-1}}$$

To see the equation for b, observe that

$$b^2 = \frac{(1-t)^2}{1-2t} = \frac{(1-\frac{1}{n+1})^2}{1-\frac{2}{n+1}} = \frac{(\frac{n}{n+1})^2}{\frac{n-1}{n+1}} = \frac{n^2}{n^2-1}$$

The Easy Case

Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

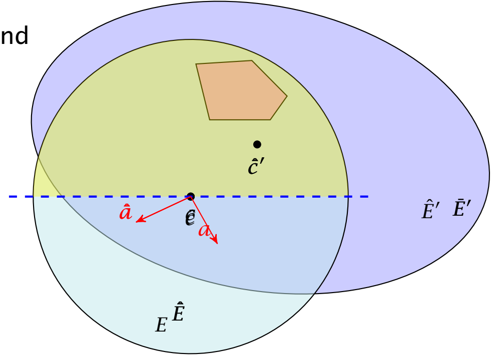
$$\begin{aligned}\gamma_n^2 &= \Big(\frac{n}{n+1}\Big)^2\Big(\frac{n^2}{n^2-1}\Big)^{n-1}\\ &= \Big(1-\frac{1}{n+1}\Big)^2\Big(1+\frac{1}{(n-1)(n+1)}\Big)^{n-1}\\ &\le e^{-2\frac{1}{n+1}}\cdot e^{\frac{1}{n+1}}\\ &= e^{-\frac{1}{n+1}}\end{aligned}$$

where we used $(1+x)^a \le e^{ax}$ for $x \in \mathbb{R}$ and $a > 0$.

This gives $\gamma_n \le e^{-\frac{1}{2(n+1)}}$.

How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
- Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1.
- Compute the new center $\hat{c}'$ and the new matrix $\hat{Q}'$ for this simplified setting.
- Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E.



Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} &\geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\ &= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^t(x - c) \leq 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^t(x - c) \leq 0\} \\ &= \{f^{-1}(f(y)) \mid a^t(f(y) - c) \leq 0\} \\ &= \{y \mid a^t(f(y) - c) \leq 0\} \\ &= \{y \mid a^t(Ly + c - c) \leq 0\} \\ &= \{y \mid (a^tL)y \leq 0\} \end{aligned}$$

This means $\bar{a} = L^t a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^t a}{\|L^t a\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^t a}{\|L^t a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^t a}{\|L^t a\|}$$

$$\begin{aligned} c' &= f(\bar{c}') = L \cdot \bar{c}' + c \\ &= -\frac{1}{n+1} L \frac{L^t a}{\|L^t a\|} + c \\ &= c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^t Q a}} \end{aligned}$$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1 e_1^t\Big)$$

because for $a = {}^{n}/_{n+1}$ and $b = {}^{n}/_{\sqrt{n^2-1}}$

$$\begin{aligned} b^2 - b^2\frac{2}{n+1} &= \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2} \\ &= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2 \end{aligned}$$

9 The Ellipsoid Algorithm

$$
\begin{aligned}
\bar{E}' &= R(\hat{E}') \\
&= \{R(x) \mid x^t \hat{Q}'^{-1} x \le 1\} \\
&= \{y \mid (R^{-1}y)^t \hat{Q}'^{-1} R^{-1} y \le 1\} \\
&= \{y \mid y^t (R^t)^{-1} \hat{Q}'^{-1} R^{-1} y \le 1\} \\
&= \{y \mid y^t (\underbrace{R\hat{Q}'R^t}_{\bar{Q}'})^{-1} y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$
\begin{aligned}
\bar{Q}' &= R\hat{Q}'R^t \\
&= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^t\Big) \cdot R^t \\
&= \frac{n^2}{n^2-1}\Big(R \cdot R^t - \frac{2}{n+1}(Re_1)(Re_1)^t\Big) \\
&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^t aa^t L}{\|L^t a\|^2}\Big)
\end{aligned}
$$

9 The Ellipsoid Algorithm

$$
\begin{aligned}
E' &= L(\bar{E}') \\
&= \{L(x) \mid x^t \bar{Q}'^{-1} x \le 1\} \\
&= \{y \mid (L^{-1}y)^t \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^t (L^t)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^t (\underbrace{L\bar{Q}'L^t}_{Q'})^{-1} y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}Q' &= L\bar{Q}'L^t\\ &= L\cdot\frac{n^2}{n^2-1}\Big(I-\frac{2}{n+1}\frac{L^taa^tL}{a^tQa}\Big)\cdot L^t\\ &= \frac{n^2}{n^2-1}\Big(Q-\frac{2}{n+1}\frac{Qaa^tQ}{a^tQa}\Big)\end{aligned}$$

Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

```
input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
output: point $x \in K$ or "$K$ is empty"
$Q \leftarrow$ ???
repeat
    if $c \in K$ then return $c$
    else
        choose a violated hyperplane $a$
        $c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^t Q a}}$
        $Q \leftarrow \frac{n^2}{n^2-1} \Big( Q - \frac{2}{n+1} \frac{Qaa^tQ}{a^tQa} \Big)$
    endif
until ???
return "$K$ is empty"
```

Repeat: Size of basic solutions

Lemma 37

Let $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ be a bounded polytop. Let $\langle a_{\max} \rangle$ be the maximum encoding length of an entry in A, b. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \le 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

In the following we use $\delta := 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

Note that here we have $P = \{x \mid Ax \le b\}$. The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.

Repeat: Size of basic solutions

Proof:

Let $\bar{A} = \begin{bmatrix} A & -A & \multirow{2}{*}{I_m} \\ -A & A & \end{bmatrix}$, $\bar{b} = \begin{pmatrix} b \\ -b \end{pmatrix}$, be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices $\bar{A}_B$ and $\bar{M}_j$ (matrix obt. when replacing the j-th column of $\bar{A}_B$ by $\bar{b}$) can become at most

$$\begin{aligned} \det(\bar{A}_B), \det(\bar{M}_j) &\le \|\vec{\ell}_{\max}\|^{2n} \\ &\le (\sqrt{2n} \cdot 2^{\langle a_{\max} \rangle})^{2n} \le 2^{2n\langle a_{\max}\rangle + 2n\log_2 n} , \end{aligned}$$

where $\vec{\ell}_{\max}$ is the longest column-vector that can be obtained after deleting all but $2n$ rows and columns from $\bar{A}$.

This holds because columns from I_m selected when going from $\bar{A}$ to $\bar{A}_B$ do not increase the determinant. Only the at most $2n$ columns from matrices A and $-A$ that $\bar{A}$ consists of contribute.

How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \le \delta$.

Hence, P is contained in the cube $-\delta \le x_i \le \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n B(0, 1) \le (n\delta)^n B(0, 1)$.

When can we terminate?

Let $P := \{x \mid Ax \leq b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop. Let $\langle a_{\max} \rangle$ be the encoding length of the largest entry in A or b.

Consider the following polytope

$$P_\lambda := \left\{ x \mid Ax \leq b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\} ,$$

where $\lambda = \delta^2 + 1$.

Lemma 38

P_λ is feasible if and only if P is feasible.

$\Leftarrow$: obvious!

$\Rightarrow$:

Consider the polytops

$$\bar{P} = \left\{x \mid \begin{bmatrix} A & -A & I_m \\ -A & A & \end{bmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix}; x \geq 0\right\}$$

and

$$\bar{P}_\lambda = \left\{x \mid \begin{bmatrix} A & -A & I_m \\ -A & A & \end{bmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}; x \geq 0\right\} \text{ .}$$

P is feasible if and only if $\bar{P}$ is feasible, and P_λ feasible if and only if $\bar{P}_\lambda$ feasible.

$\bar{P}_\lambda$ is bounded since P_λ and P are bounded.

Let $\bar{A} = \begin{bmatrix} A & -A & \\ -A & A & I_m \end{bmatrix}$, and $\bar{b} = \begin{pmatrix} b \\ -b \end{pmatrix}$.

$\bar{P}_\lambda$ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1}\bar{b} + \frac{1}{\lambda}\bar{A}_B^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x-values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

$$(\bar{A}_B^{-1}\bar{b})_i < 0 \le (\bar{A}_B^{-1}\bar{b})_i + \frac{1}{\lambda}(\bar{A}_B^{-1}\vec{1})_i$$

By Cramers rule we get

$$(\bar{A}_B^{-1}\bar{b})_i < 0 \quad \Longrightarrow \quad (\bar{A}_B^{-1}\bar{b})_i \le -\frac{1}{\det(\bar{A}_B)}$$

and

$$(\bar{A}_B^{-1}\vec{1})_i \le \det(\bar{M}_j) \ ,$$

where $\bar{M}_j$ is obtained by replacing the j-th column of $\bar{A}_B$ by $\vec{1}$.

However, we showed that the determinants of $\bar{A}_B$ and $\bar{M}_j$ can become at most δ.

Since, we chose $\lambda = \delta^2 + 1$ this gives a contradiction.

Lemma 39

If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \mathrm{vol}(B(0,1)) = \frac{1}{\delta^{3n}}\mathrm{vol}(B(0,1))$.

Proof:

If P_λ feasible then also P. Let x be feasible for P.
This means $Ax \le b$.

Let $\vec{\ell}$ with $\|\vec{\ell}\| \le r$. Then

$$\begin{aligned}(A(x+\vec{\ell}))_i &= (Ax)_i + (A\vec{\ell})_i \le b_i + A_i\vec{\ell} \\ &\le b_i + \|A_i\| \cdot \|\vec{\ell}\| \le b_i + \sqrt{n} \cdot 2^{\langle a_{\max}\rangle} \cdot r \\ &\le b_i + \frac{\sqrt{n} \cdot 2^{\langle a_{\max}\rangle}}{\delta^3} \le b_i + \frac{1}{\delta^2+1} \le b_i + \frac{1}{\lambda}\end{aligned}$$

Hence, $x + \vec{\ell}$ is feasible for P_λ which proves the lemma.

How many iterations do we need until the volume becomes too small?

$$e^{-\frac{i}{2(n+1)}} \cdot \mathrm{vol}(B(0,R)) < \mathrm{vol}(B(0,r))$$

Hence,

$$\begin{aligned} i &> 2(n+1)\ln\Big(\frac{\mathrm{vol}(B(0,R))}{\mathrm{vol}(B(0,r))}\Big) \\ &= 2(n+1)\ln\Big(n^n\delta^n \cdot \delta^{3n}\Big) \\ &= 8n(n+1)\ln(\delta) + 2(n+1)n\ln(n) \\ &= \mathcal{O}(\mathrm{poly}(n, \langle a_{\max}\rangle)) \end{aligned}$$

Algorithm 1 ellipsoid-algorithm

```
input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii $R$ and $r$
        with $K \subseteq B(c,R)$, and $B(x,r) \subseteq K$ for some $x$
output: point $x \in K$ or "$K$ is empty"
$Q \leftarrow \mathrm{diag}(R^2,\dots,R^2)$ // i.e., $L = \mathrm{diag}(R,\dots,R)$
repeat
    if $c \in K$ then return $c$
    else
        choose a violated hyperplane $a$
        $c \leftarrow c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^tQa}}$
        $Q \leftarrow \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Qaa^tQ}{a^tQa}\Big)$
    endif
until $\det(Q) \le r^{2n}$ // i.e., $\det(L) \le r^n$
return "$K$ is empty"
```

Separation Oracle:

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- certifies that $x \in K$,
- or finds a hyperplane separating x from K.

We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- a guarantee that a ball of radius r is contained in K,
- an initial ball $B(c, R)$ with radius R that contains K,
- a separation oracle for K.

The Ellipsoid algorithm requires $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

10 Karmarkars Algorithm

We want to solve the following linear program:

- $\min v = c^t x$ subject to $Ax = 0$ and $x \in \Delta$.
- Here $\Delta = \{x \in \mathbb{R}^n \mid e^t x = 1, x \geq 0\}$ with $e^t = (1, \dots, 1)$ denotes the standard simplex in $\mathbb{R}^n$.

Further assumptions:

1. A is an $m \times n$-matrix with rank m.
2. $Ae = 0$, i.e., the center of the simplex is feasible.
3. The optimum solution is 0.

10 Karmarkars Algorithm

Suppose you start with $\max\{c^t x \mid Ax = b; x \geq 0\}$.

- Multiply c by -1 and do a minimization. ⇒ minimization problem
- We can check for feasibility by using the two phase algorithm. ⇒ can assume that LP is feasible.
- Compute the dual; pack primal and dual into one LP and minimize the duality gap. ⇒ optimum is 0
- Add a new variable pair x_ℓ, x'_ℓ (both restricted to be positive) and the constraint $\sum_i x_i = 1$. ⇒ solution in simplex
- Add $-(\sum_i x_i)b_i = -b_i$ to every constraint. ⇒ vector b is 0
- If A does not have full row rank we can delete constraints (or conclude that the LP is infeasible).
  ⇒ A has full row rank

We still need to make e/n feasible.

10 Karmarkars Algorithm

The algorithm computes strictly feasible interior points $x^{(0)} = \frac{e}{n}, x^{(1)}, x^{(2)}, \ldots$ with

$$c^t x^{(k)} \leq 2^{-\Theta(L)} c^t x^{(0)}$$

For $k = \Theta(L)$. A point x is strictly feasible if $x > 0$.

If my objective value is close enough to 0 (the optimum!!) I can "snap" to an optimum vertex.

10 Karmarkars Algorithm

Iteration:

1. Distort the problem by mapping the simplex onto itself so that the current point $\bar{x}$ moves to the center.
2. Project the optimization direction c onto the feasible region. Determine a distance to travel along this direction such that you do not leave the simplex (and you do not touch the border). $\hat{x}_{\text{new}}$ is the point you reached.
3. Do a backtransformation to transform $\hat{x}$ into your new point $\bar{x}_{\text{new}}$.

The Transformation

Let $\bar{Y} = \mathrm{diag}(\bar{x})$ the diagonal matrix with entries $\bar{x}$ on the diagonal.

Define

$$F_{\bar{x}} : x \mapsto \frac{\bar{Y}^{-1}x}{e^t\bar{Y}^{-1}x} \enspace .$$

The inverse function is

$$F_{\bar{x}}^{-1} : \hat{x} \mapsto \frac{\bar{Y}\hat{x}}{e^t\bar{Y}\hat{x}} \enspace .$$

Note that $\bar{x} > 0$ in every coordinate. Therefore the above is well defined.

Properties

$F_{\bar{x}}^{-1}$ really is the inverse of $F_{\bar{x}}$:

$$F_{\bar{x}}(F_{\bar{x}}^{-1}(\hat{x})) = \frac{\bar{Y}^{-1}\frac{\bar{Y}\hat{x}}{e^t\bar{Y}\hat{x}}}{e^t\bar{Y}^{-1}\frac{\bar{Y}\hat{x}}{e^t\bar{Y}\hat{x}}} = \frac{\hat{x}}{e^t\hat{x}} = \hat{x}$$

because $\hat{x} \in \Delta$.

Note that in particular every $\hat{x} \in \Delta$ has a preimage (Urbild) under $F_{\bar{x}}$.

Properties

$\bar{x}$ is mapped to e/n

$$F_{\bar{x}}(\bar{x}) = \frac{\bar{Y}^{-1}\bar{x}}{e^t\bar{Y}^{-1}\bar{x}} = \frac{e}{e^te} = \frac{e}{n}$$

Properties

A unit vectors e_i is mapped to itself:

$$F_{\bar{x}}(e_i) = \frac{\bar{Y}^{-1}e_i}{e^t\bar{Y}^{-1}e_i} = \frac{(0,\dots,0,1/\bar{x}_i,0,\dots,0)^t}{e^t(0,\dots,0,1/\bar{x}_i,0,\dots,0)^t} = e_i$$

Properties

All nodes of the simplex are mapped to the simplex:

$$F_{\bar{x}}(x) = \frac{\bar{Y}^{-1}x}{e^t\bar{Y}^{-1}x} = \frac{\left(\frac{x_1}{\bar{x}_1},\dots,\frac{x_n}{\bar{x}_n}\right)^t}{e^t\left(\frac{x_1}{\bar{x}_1},\dots,\frac{x_n}{\bar{x}_n}\right)^t} = \frac{\left(\frac{x_1}{\bar{x}_1},\dots,\frac{x_n}{\bar{x}_n}\right)^t}{\sum_i \frac{x_i}{\bar{x}_i}} \in \Delta$$

The Transformation

Easy to check:

- $F_{\bar{x}}^{-1}$ really is the inverse of $F_{\bar{x}}$.
- $\bar{x}$ is mapped to e/n.
- A unit vectors e_i is mapped to itself.
- All nodes of the simplex are mapped to the simplex.

10 Karmarkars Algorithm

We have the problem

$$\begin{aligned}
&\min\{c^t x \mid Ax = 0;\, x \in \Delta\} \\
&\quad = \min\{c^t F_{\bar{x}}^{-1}(\hat{x}) \mid AF_{\bar{x}}^{-1}(\hat{x}) = 0;\, F_{\bar{x}}^{-1}(\hat{x}) \in \Delta\} \\
&\quad = \min\{c^t F_{\bar{x}}^{-1}(\hat{x}) \mid AF_{\bar{x}}^{-1}(\hat{x}) = 0;\, \hat{x} \in \Delta\} \\
&\quad = \min\left\{\frac{c^t \bar{Y}\hat{x}}{e^t \bar{Y}\hat{x}} \mid \frac{A\bar{Y}\hat{x}}{e^t \bar{Y}\hat{x}} = 0;\, \hat{x} \in \Delta\right\}
\end{aligned}$$

Since the optimum solution is 0 this problem is the same as

$$\min\{\hat{c}^t \hat{x} \mid \hat{A}\hat{x} = 0, \hat{x} \in \Delta\}$$

with $\hat{c} = \bar{Y}^t c = \bar{Y} c$ and $\hat{A} = A\bar{Y}$.

Note that $e^t \bar{Y} x > 0$ for $x \in \Delta$.

We still need to make e/n feasible.

- We know that our LP is feasible. Let $\bar{x}$ be a feasible point.
- Apply $F_{\bar{x}}$, and solve

$$\min\{\hat{c}^t x \mid \hat{A}x = 0; x \in \Delta\}$$

- The feasible point is moved to the center.

10 Karmarkars Algorithm

When computing $\hat{x}_{\text{new}}$ we do not want to leave the simplex or touch its boundary (why?).

For this we compute the radius of a ball that completely lies in the simplex.

$$B\left(\frac{e}{n},\rho\right) = \left\{x \in \mathbb{R}^n \mid \left\|x - \frac{e}{n}\right\| \le \rho\right\} \quad .$$

We are looking for the largest radius r such that

$$B\left(\frac{e}{n},r\right) \cap \left\{x \mid e^t x = 1\right\} \subseteq \Delta.$$

10 Karmarkars Algorithm

This holds for $r = \|\frac{e}{n} - (e - e_1)\frac{1}{n-1}\|$. ($r$ is the distance between the center e/n and the center of the $(n-1)$-dimensional simplex obtained by intersecting a side ($x_i = 0$) of the unit cube with Δ.)

This gives $r = \frac{1}{\sqrt{n(n-1)}}$.

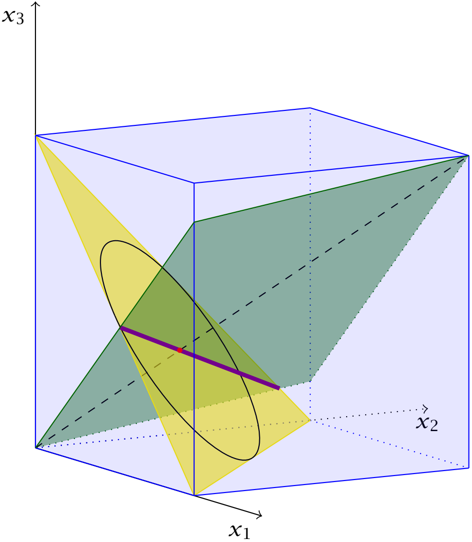
Now we consider the problem

$$\min\{\hat{c}^t x \mid \hat{A}x = 0, x \in B(e/n, r) \cap \Delta\}$$

This problem is easy to solve!!!

$$r^2 = (n-1) \cdot \left(\frac{1}{n} - \frac{1}{n-1}\right)^2 + \frac{1}{n^2} = \frac{1}{n^2(n-1)} + \frac{1}{n^2} = \frac{1}{n(n-1)}$$

The Simplex



10 Karmarkars Algorithm

Ideally we would like to go in direction of $-\hat{c}$ (starting from the center of the simplex).

However, doing this may violate constraints $\hat{A}\hat{x} = 0$ or the constraint $\hat{x} \in \Delta$.

Therefore we first project $\hat{c}$ on the nullspace of

$$B = \begin{pmatrix} \hat{A} \\ e^t \end{pmatrix}$$

We use

$$P = I - B^t(BB^t)^{-1}B$$

Then

$$\hat{d} = P\hat{c}$$

is the required projection.

10 Karmarkars Algorithm

We get the new point

$$\hat{x}(\rho) = \frac{e}{n} - \rho \frac{\hat{d}}{\|\hat{d}\|}$$

for $\rho < r$.

Choose $\rho = \alpha r$ with $\alpha = 1/4$.

Iteration of Karmarkars Algorithm

- Current solution $\bar{x}$. $\bar{Y} := \mathrm{diag}(\bar{x}_1, \dots, \bar{x}_n)$.
- Transform problem via $F_{\bar{x}}(x) = \frac{\bar{Y}^{-1}x}{e^t\bar{Y}^{-1}x}$.
  Let $\hat{c} = \bar{Y}c$, and $\hat{A} = A\bar{Y}$.
- Compute
  $$\hat{d} = (I - B^t(BB^t)^{-1}B)\hat{c} \ ,$$
  where $B = \begin{pmatrix} \hat{A} \\ e^t \end{pmatrix}$.
- Set
  $$\hat{x}_{\text{new}} = \frac{e}{n} - \rho\frac{\hat{d}}{\|\hat{d}\|} \ ,$$
  with $\rho = \alpha r$ with $\alpha = 1/4$ and $r = 1/\sqrt{n(n-1)}$.
- Compute $\bar{x}_{\text{new}} = F_{\bar{x}}^{-1}(\hat{x}_{\text{new}})$.

Lemma 40

The new point $\hat{x}_{\text{new}}$ *in the transformed space is the point that minimizes the cost* $\hat{c}^t\hat{x}$ *among all feasible points in* $B(\frac{e}{n},\rho)$*.*

Proof: Let $\hat{z}$ be another feasible point in $B(\frac{e}{n}, \rho)$.

As $\hat{A}\hat{z} = 0$, $\hat{A}\hat{x}_{\text{new}} = 0$, $e^t\hat{z} = 1$, $e^t\hat{x}_{\text{new}} = 1$ we have

$$B(\hat{x}_{\text{new}} - \hat{z}) = 0 \ .$$

Further,

$$\begin{aligned}(\hat{c} - \hat{d})^t &= (\hat{c} - P\hat{c})^t \\ &= (B^t(BB^t)^{-1}B\hat{c})^t \\ &= \hat{c}^t B^t(BB^t)^{-1}B\end{aligned}$$

Hence, we get

$$(\hat{c} - \hat{d})^t(\hat{x}_{\text{new}} - \hat{z}) = 0 \text{ or } \hat{c}^t(\hat{x}_{\text{new}} - \hat{z}) = \hat{d}^t(\hat{x}_{\text{new}} - \hat{z})$$

which means that the cost-difference between $\hat{x}_{\text{new}}$ and $\hat{z}$ is the same measured w.r.t. the cost-vector $\hat{c}$ or the projected cost-vector $\hat{d}$.

But

$$\frac{\hat{d}^t}{\|\hat{d}\|}\,(\hat{x}_{\text{new}} - \hat{z}) \;=\; \frac{\hat{d}^t}{\|\hat{d}\|}\left(\frac{e}{n} - \rho\frac{\hat{d}}{\|\hat{d}\|} - \hat{z}\right) \;=\; \frac{\hat{d}^t}{\|\hat{d}\|}\left(\frac{e}{n} - \hat{z}\right) - \rho < 0$$

as $\frac{e}{n} - \hat{z}$ is a vector of length at most ρ.

This gives $\hat{d}(\hat{x}_{\text{new}} - \hat{z}) \le 0$ and therefore $\hat{c}\hat{x}_{\text{new}} \le \hat{c}\hat{z}$.

In order to measure the progress of the algorithm we introduce a potential function f:

$$f(x)=\sum_j \ln(\frac{c^tx}{x_j})=n\ln(c^tx)-\sum_j \ln(x_j)\enspace.$$

- The function f is invariant to scaling (i.e., $f(kx)=f(x)$).
- The potential function essentially measures cost (note the term $n\ln(c^tx)$) but it penalizes us for choosing x_j values very small (by the term $-\sum_j \ln(x_j)$; note that $-\ln(x_j)$ is always positive).

For a point $\hat{z}$ in the transformed space we use the potential function

$$\hat{f}(\hat{z}) := f(F_{\bar{x}}^{-1}(\hat{z})) = f(\frac{\bar{Y}\hat{z}}{e^t\bar{Y}\hat{z}}) = f(\bar{Y}\hat{z})$$
$$= \sum_j \ln(\frac{c^t\bar{Y}\hat{z}}{\bar{x}_j\hat{z}_j}) = \sum_j \ln(\frac{\hat{c}^t\hat{z}}{\hat{z}_j}) - \sum_j \ln \bar{x}_j$$

Observation:
This means the potential of a point in the transformed space is simply the potential of its pre-image under F.

Note that if we are interested in potential-change we can ignore the additive term above. Then f and $\hat{f}$ have the same form; only c is replaced by $\hat{c}$.

The basic idea is to show that one iteration of Karmarkar results in a constant decrease of $\hat{f}$. This means

$$\hat{f}(\hat{x}_{\text{new}}) \le \hat{f}(\frac{e}{n}) - \delta \ ,$$

where δ is a constant.

This gives

$$f(\bar{x}_{\text{new}}) \le f(\bar{x}) - \delta \ .$$

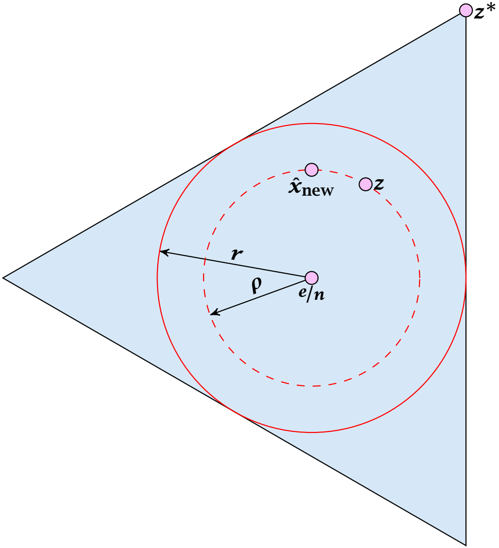
Lemma 41

There is a feasible point z (i.e., $\hat{A}z = 0$) in $B(\frac{e}{n}, \rho) \cap \Delta$ that has

$$\hat{f}(z) \le \hat{f}(\frac{e}{n}) - \delta$$

with $\delta = \ln(1 + \alpha)$.

Note that this shows the existence of a good point within the ball. In general it will be difficult to find this point.


z^*
$\hat{x}_{\text{new}}$
z
r
ρ
e/n

Let z^* be the feasible point in the transformed space where $\hat{c}^t x$ is minimized. (Note that in contrast $\hat{x}_{\text{new}}$ is the point in the intersection of the feasible region and $B(\frac{e}{n}, \rho)$ that minimizes this function; in general $z^* \neq \hat{x}_{\text{new}}$)

z^* must lie at the boundary of the simplex. This means $z^* \notin B(\frac{e}{n}, \rho)$.

The point z we want to use lies farthest in the direction from $\frac{e}{n}$ to z^*, namely

$$z = (1 - \lambda)\frac{e}{n} + \lambda z^*$$

for some positive $\lambda < 1$.

Hence,

$$\hat{c}^t z = (1-\lambda)\hat{c}^t \frac{e}{n} + \lambda \hat{c}^t z^*$$

The optimum cost (at z^*) is zero.

Therefore,

$$\frac{\hat{c}^t \frac{e}{n}}{\hat{c}^t z} = \frac{1}{1-\lambda}$$

The improvement in the potential function is

$$
\begin{aligned}
\hat{f}(\frac{e}{n}) - \hat{f}(z) &= \sum_j \ln(\frac{\hat{c}^t \frac{e}{n}}{\frac{1}{n}}) - \sum_j \ln(\frac{\hat{c}^t z}{z_j}) \\
&= \sum_j \ln(\frac{\hat{c}^t \frac{e}{n}}{\hat{c}^t z} \cdot \frac{z_j}{\frac{1}{n}}) \\
&= \sum_j \ln(\frac{n}{1-\lambda} z_j) \\
&= \sum_j \ln(\frac{n}{1-\lambda}((1-\lambda)\frac{1}{n} + \lambda z_j^*)) \\
&= \sum_j \ln(1 + \frac{n\lambda}{1-\lambda} z_j^*)
\end{aligned}
$$

We can use the fact that for non-negative s_i

$$\boldsymbol{\sum_i \ln(1 + s_i) \geq \ln(1 + \sum_i s_i)}$$

This gives

$$\begin{aligned}\hat{f}(\frac{e}{n}) - \hat{f}(z) &= \sum_j \ln(1 + \frac{n\lambda}{1-\lambda} z_j^*) \\ &\geq \ln(1 + \frac{n\lambda}{1-\lambda})\end{aligned}$$

Suppose true for $s_1, \dots, s_{k-1}$. Then

$$\begin{aligned}\sum_{i=1}^{k} \ln(1 + s_i) \geq \ln(1 + \textstyle\sum_{i=1}^{k-1} s_i) + \ln(1 + s_k) &= \ln\Big((1 + \textstyle\sum_{i=1}^{k-1} s_i)(1 + s_k)\Big) \\ &= \ln\Big(1 + \sum_i s_i + s_k \textstyle\sum_{i=1}^{k-1} s_i\Big) \geq \ln(1 + \textstyle\sum_i s_i)\end{aligned}$$

In order to get further we need a bound on λ:

$$\alpha r = \rho = \|z - {}^{e}/_{n}\| = \|\lambda(z^* - {}^{e}/_{n})\| \leq \lambda R$$

Here R is the radius of the ball around $\frac{e}{n}$ that contains the whole simplex.

$R = \sqrt{(n-1)/n}$. Since $r = {}^{1}/_{\sqrt{(n-1)n}}$ we have $R/r = n - 1$ and

$$\lambda \geq \alpha\frac{r}{R} \geq \alpha/(n-1)$$

Then

$$1 + n\frac{\lambda}{1-\lambda} \geq 1 + \frac{n\alpha}{n-\alpha-1} \geq 1 + \alpha$$

This gives the lemma.

Lemma 42

If we choose $\alpha = 1/4$ *and* $n \geq 4$ *in Karmarkars algorithm the point* $\hat{x}_{\text{new}}$ *satisfies*

$$\hat{f}(\hat{x}_{\text{new}}) \leq \hat{f}(\frac{e}{n}) - \delta$$

with $\delta = 1/10$*.*

Proof:

Define

$$\begin{aligned} g(\hat{x}) &= n \ln \frac{\hat{c}^t \hat{x}}{\hat{c}^t \frac{e}{n}} \\ &= n(\ln \hat{c}^t \hat{x} - \ln \hat{c}^t \frac{e}{n}) \enspace . \end{aligned}$$

This is the change in the cost part of the potential function when going from the center $\frac{e}{n}$ to the point $\hat{x}$ in the transformed space.

Similar, the penalty when going from $\frac{e}{n}$ to w increases by

$$h(\hat{x}) = \mathrm{pen}(\hat{x}) - \mathrm{pen}(\frac{e}{n}) = -\sum_j \ln \frac{\hat{x}_j}{\frac{1}{n}}$$

where $\mathrm{pen}(v) = -\sum_j \ln(v_j)$.

We want to derive a lower bound on

$$\begin{aligned}\hat{f}(\frac{e}{n}) - \hat{f}(\hat{x}_{\text{new}}) = {} & [\hat{f}(\frac{e}{n}) - \hat{f}(z)] \\ & + h(z) \\ & - h(\hat{x}_{\text{new}}) \\ & + [g(z) - g(\hat{x}_{\text{new}})]\end{aligned}$$

where z is the point in the ball where $\hat{f}$ achieves its minimum.

We have

$$[\hat{f}(\frac{e}{n}) - \hat{f}(z)] \geq \ln(1 + \alpha)$$

by the previous lemma.

We have

$$[g(z) - g(\hat{x}_{\text{new}})] \geq 0$$

since $\hat{x}_{\text{new}}$ is the point with minimum cost in the ball, and g is monotonically increasing with cost.

We show that the change $h(w)$ in penalty when going from e/n to w fulfills

$$|h(w)| \leq \frac{\beta^2}{2(1-\beta)}$$

where $\beta = n\alpha r$ and w is some point in the ball $B(\frac{e}{n}, \alpha r)$.

Hence,

$$\hat{f}(\frac{e}{n}) - \hat{f}(\hat{x}_{\text{new}}) \geq \ln(1+\alpha) - \frac{\beta^2}{(1-\beta)} \ .$$

Lemma 43

For $|x| \leq \beta < 1$

$$|\ln(1+x) - x| \leq \frac{x^2}{2(1-\beta)} \ .$$

For $|x| < 1$

$$\ln(1+x) = \sum_{i \geq 1} (-1)^{i+1} \frac{x^i}{i} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

This gives

$$|\ln(1+x) - x| \leq \left| -\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right| \leq \left| \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \right|$$
$$\leq \frac{x^2}{2} \left| x^0 + x^1 + x^2 + \dots \right| = \frac{x^2}{2(1-|x|)} \ .$$

This gives for $w \in B(\frac{e}{n}, \rho)$

$$
\begin{aligned}
|h(w)| &= \left| \sum_j \ln \frac{w_j}{1/n} \right| \\
&= \left| \sum_j \ln \left(\frac{1/n + (w_j - 1/n)}{1/n} \right) - \overset{\boxed{=0}}{\sum_j n\left(w_j - \frac{1}{n}\right)} \right| \\
&= \left| \sum_j \left[\ln \left(1 + \overset{\boxed{\leq n \boxed{x} < 1}}{n(w_j - 1/n)} \right) - \overset{\boxed{x}}{n(w_j - 1/n)} \right] \right| \\
&\leq \sum_j \frac{n^2(w_j - 1/n)^2}{2(1 - \alpha n r)} \\
&\leq \frac{(\alpha n r)^2}{2(1 - \alpha n r)}
\end{aligned}
$$

The decrease in potential is therefore at least

$$\ln(1+\alpha) - \frac{\beta^2}{1-\beta}$$

with $\beta = n\alpha r = \alpha\sqrt{\frac{n}{n-1}}$.

It can be shown that this is at least $\frac{1}{10}$ for $n \geq 4$ and $\alpha = 1/4$.

Let $\bar{x}^{(k)}$ be the current point after the k-th iteration, and let $\bar{x}^{(0)} = \frac{e}{n}$.

Then $f(\bar{x}^{(k)}) \leq f(e/n) - k/10$.
This gives

$$\begin{aligned} n \ln \frac{c^t \bar{x}^{(k)}}{c^t \frac{e}{n}} &\leq \sum_j \ln \bar{x}_j^{(k)} - \sum_j \ln \frac{1}{n} - k/10 \\ &\leq n \ln n - k/10 \end{aligned}$$

Choosing $k = 10n(\ell + \ln n)$ with $\ell = \Theta(L)$ we get

$$\frac{c^t \bar{x}^{(k)}}{c^t \frac{e}{n}} \leq e^{-\ell} \leq 2^{-\ell} \ .$$

Hence, $\Theta(nL)$ iterations are sufficient. One iteration can be performed in time $\mathcal{O}(n^3)$.