Distance Labelings

Ferienakademie im Sarntal — Course 2
Distance Problems: Theory and Praxis

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Outline

1. Motivation
2. Dominating sets and collections
3. An upper bound for general graphs
4. Summary

Motivation

What is distance labeling?

Given a graph $G = (V, E)$, $G \in \mathcal{G}$ which belongs to a specific class $\mathcal{G}$ of graphs. A distance labeling $\langle L, f \rangle$ consists of

- vertex labels $L(u, G)$ for all vertices $u \in V$ and a
- distance decoder f such that

$$f(L(u, G), L(v, G)) = d(u, v) \quad \forall u, v \in V.$$

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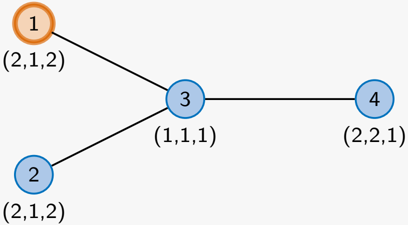

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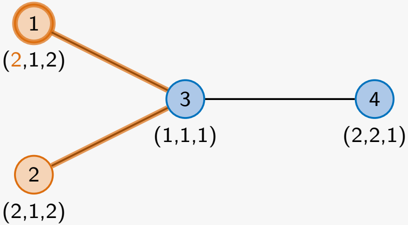

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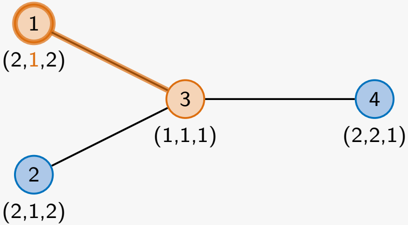

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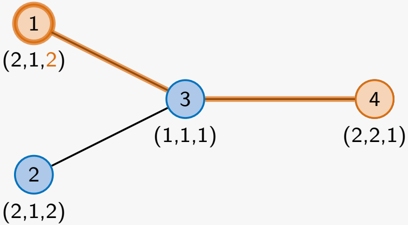

Motivation

Complexity:

For an n-vertex graph $G = (V, E)$ the previous example uses

- labels $L(u, G)$ of length $|L(u, G)| \in \mathcal{O}(|n \cdot \log n|)$
- containing the distances $d(u, v)$ to all other vertices $v \in V$
- which makes decoding possible in $\mathcal{O}(1)$.

Are better labeling schemes available?

- Label size linear in n at cost of decoding time?
- What about upper and lower bounds for label length?

Outline

Dominating set

Given a general, connected, and undirected graph $G = (V, E)$ with unit edge weights, we call $S \subseteq V$ ρ-dominating set for G if

$$\forall v \in V \;\; \exists w \in S : \; d(v, w) \leq \rho.$$

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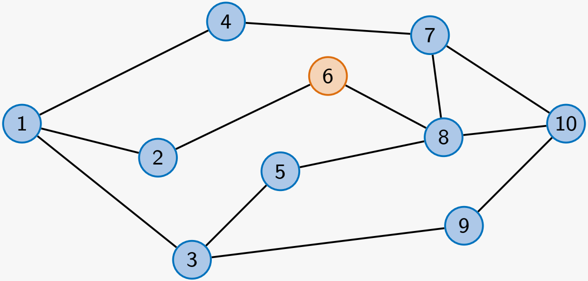

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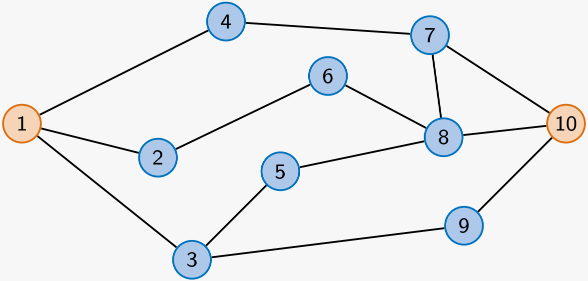

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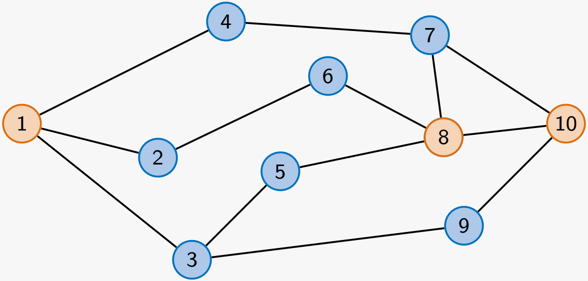

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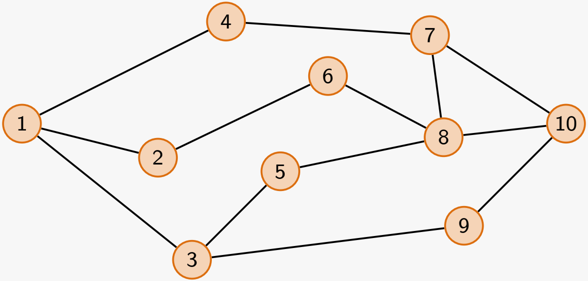

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$(S_3, \rho = 0) = V$

Dominator of a vertex

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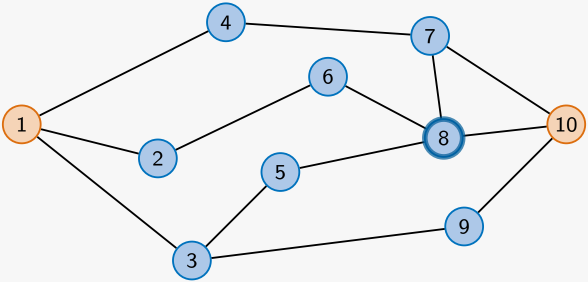

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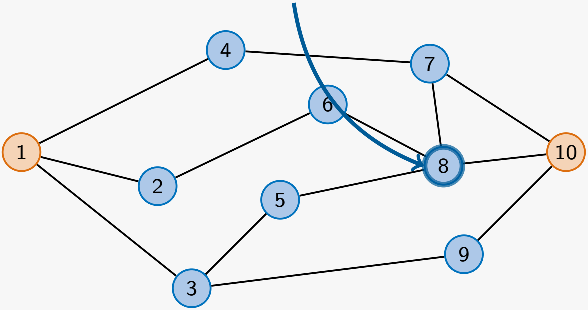
Example:

8 is dominated by 10



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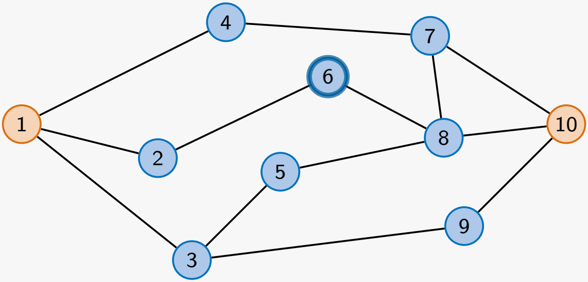

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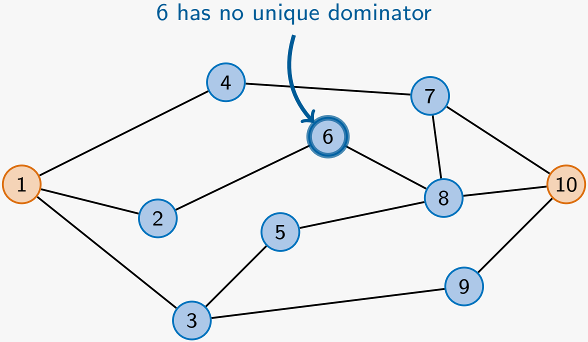

Distance between dominators

Lemma: For every two vertices $x, y \in V$ holds:

1. $d(dom_S(x), dom_S(y)) - 2\rho \leq d(x, y) \leq d(dom_S(x), dom_S(y)) + 2\rho$
2. $d(x, y)$ can be derived from
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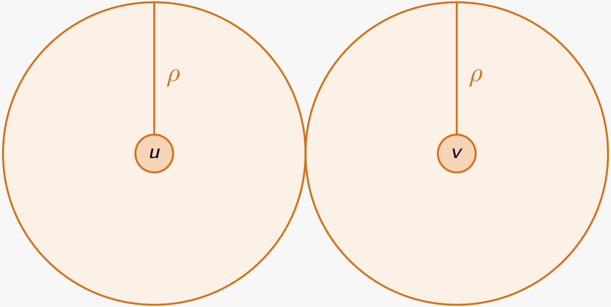

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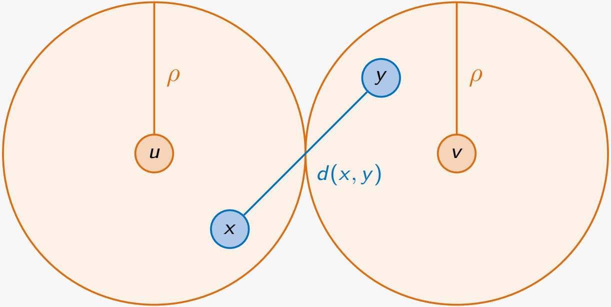

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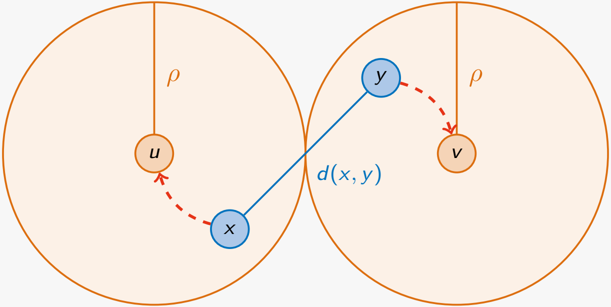

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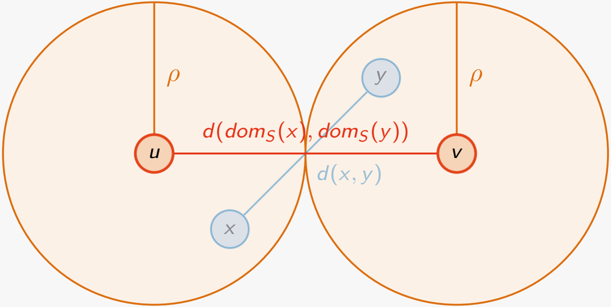

Calculating dominating sets

Finding a minimum dominating set is $\mathcal{NP}$-hard. It might be a good idea to avoid this...

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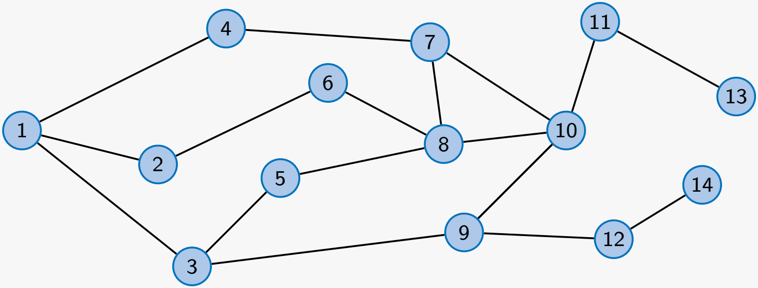
Instead, find a dominating set which is "small enough":

- Using a BFS, construct a spanning tree T on $G = (V, E)$.
- Let h denote the height of T. Divide V into disjoint sets T_i for $i \in \{0, \ldots, h\}$ according to their level in T
- Merge T_i into $\rho + 1$ disjoint sets $D_i = \bigcup_{j \in \{0,\ldots,h\}} T_{i+j(\rho+1)}$.

Calculating dominating sets

Step 1: Find the spanning tree T



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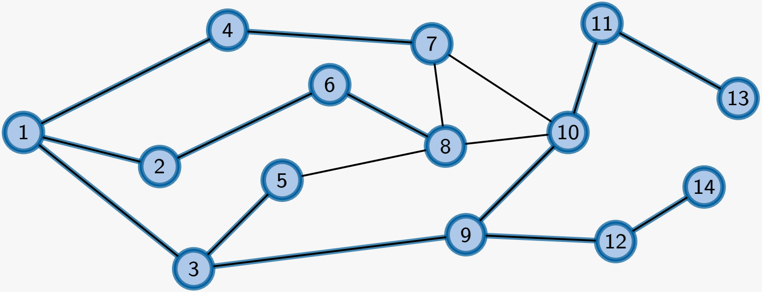

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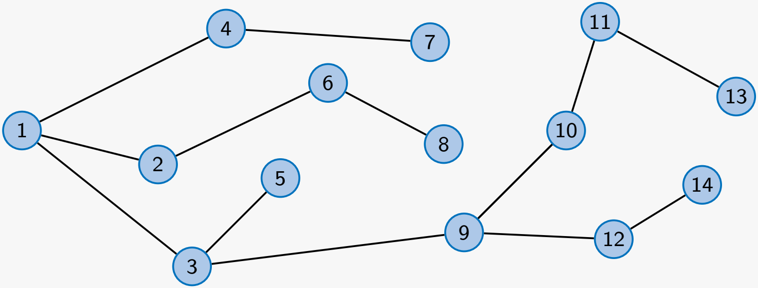

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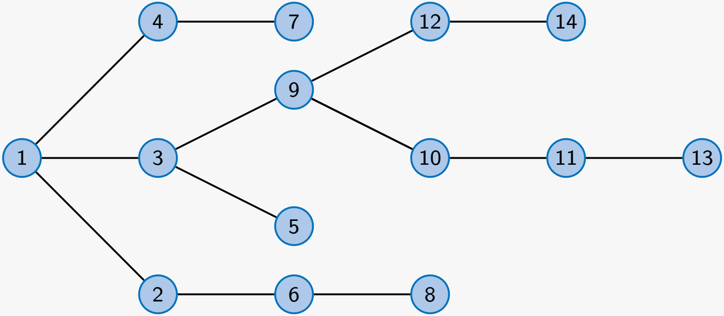
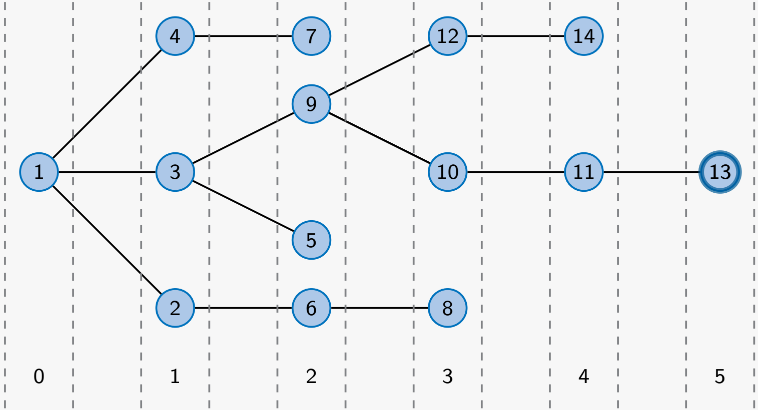


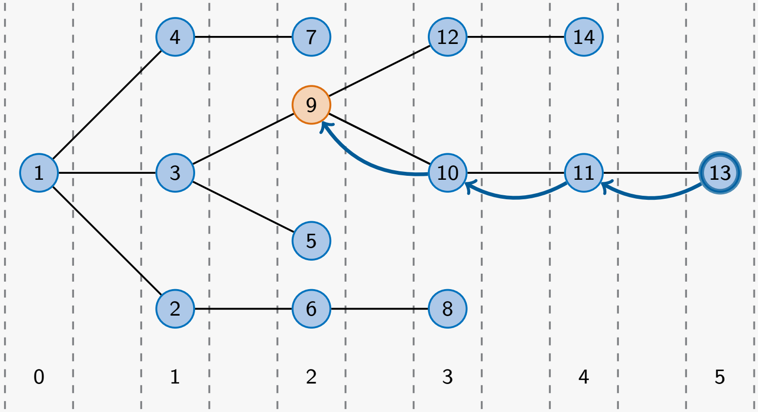
Calculating dominating sets

Step 2: Start at a not yet dominated leaf in maximum depth



Calculating dominating sets

Step 3: Go up ρ edges and add this vertex the dominating set ($\rho = 3$)



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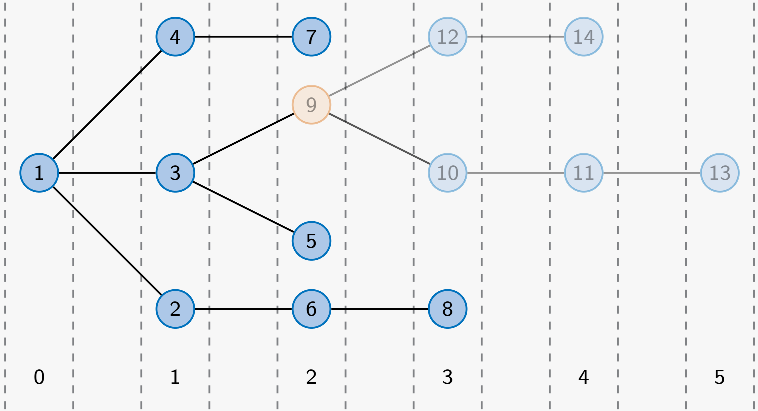
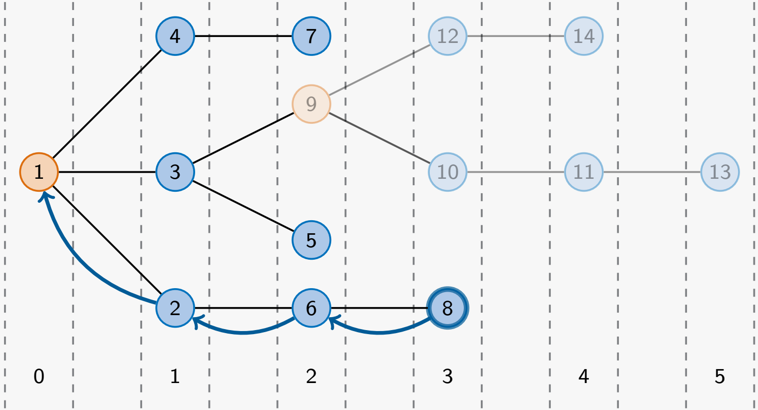


Vertex 9 dominates itself and the subtree rooted at 9 (at least $\rho + 1$ vertices)

Calculating dominating sets

Step 3: Repeat steps 2 and 3 until all vertices are dominated



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Calculating dominating sets

Size of dominating sets

Lemma: For every n-vertex connected graph G and integer $\rho \geq 0$, there exists a ρ-dominating set S such that

$$|S| \leq \max\left\{\left\lfloor\frac{n}{\rho+1}\right\rfloor, 1\right\}.$$

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Proof:

- Each element in S dominates at least $\rho + 1$ vertices in V:
  $\Rightarrow |S| \leq \left\lfloor \frac{n}{\rho+1} \right\rfloor$
- For $\rho \geq n - 1$, any single vertex $v \in V$ forms a ρ-dominating set
  $\Rightarrow |S| \geq 1$

Dominating collection

Given a decreasing sequence ρ_i with $0 \leq i \leq k$ such that $\rho_k = 0$, we call $\mathcal{S} = \{(S_i, \rho_i) \mid 0 \leq i \leq k\}$ a dominating collection.

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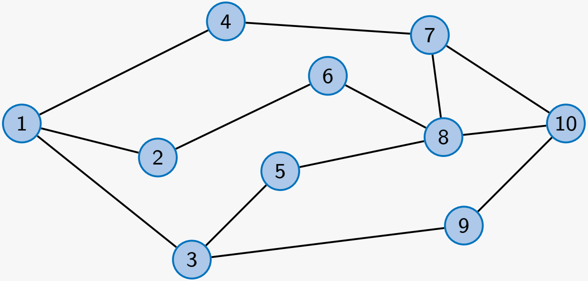

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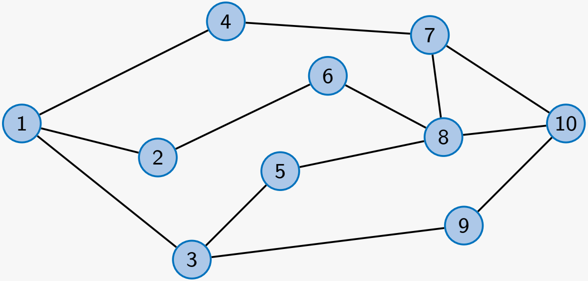

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Note: The sequence of ρ_i decreases $(\rho_k = 0)$ while $|S_i|$ increases.

Outline

General graphs

The trivial labels presented in the introduction stored the distances to all other vertices, resulting in

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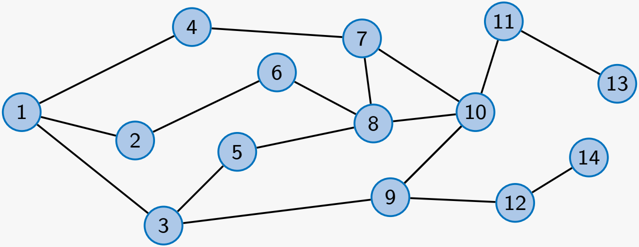
We now discuss a labeling scheme which allows for

- labels in $\mathcal{O}(n)$ and
- decoding in $\mathcal{O}(\log \log n)$.

Building the labels



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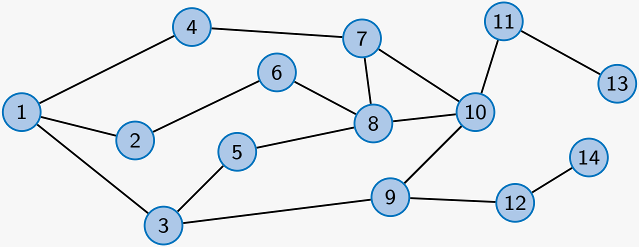
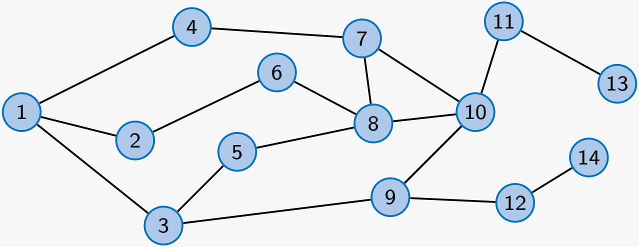


Define (don't ask why):

- $k = \lceil \log \log n \rceil$
- $I = \{0, 1, \ldots, k\}$
- $\rho_i = 2^{k-i} - 1$

Building the labels



Determine a dominating collection $\mathcal{S} = \{(S_i, \rho_i) \mid i \in I\}$:

- $\rho_0 = 3 \;\Rightarrow$
- $\rho_1 = 1 \;\Rightarrow$
- $\rho_2 = 0 \;\Rightarrow$

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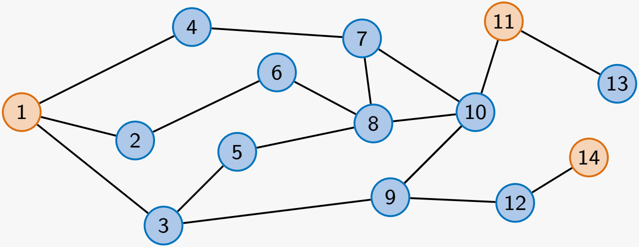

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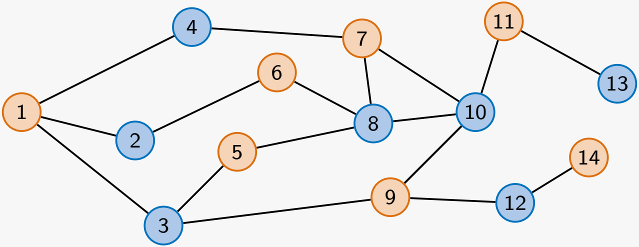

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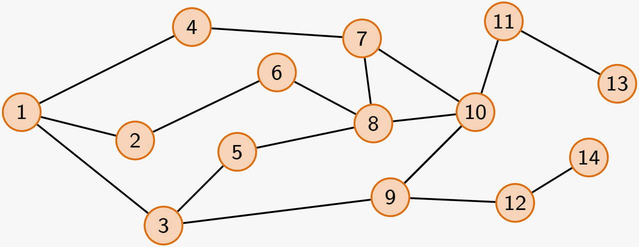

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- $\rho_1 = 1 \;\Rightarrow\; S_1 = \{1, 5, 6, 7, 9, 11, 14\}$
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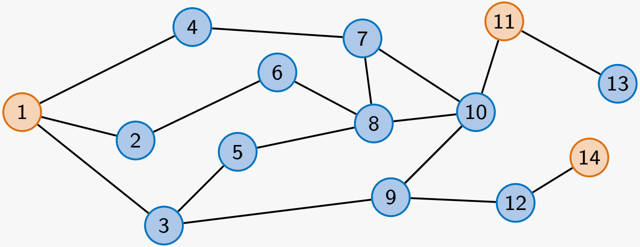

Step 0 (initialization): $i = 0$, $S_0 = \{1, 11, 14\}$, $\rho_0 = 3$

Create labels $L^0(u)$ for all $u \in S_0$ consisting of two fields containing

- the rank order(u) in the ordering of S_0 and
- the list $\{d(u, v)\}_{v \in S_0}$ according to the ordering of S_0.

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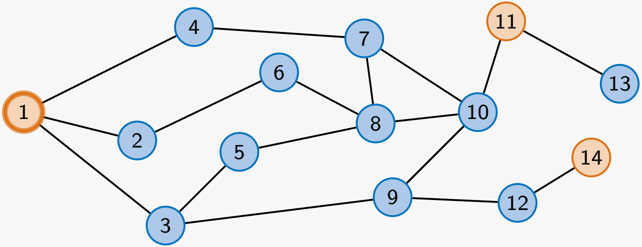

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$$L^0(1) = (0, 0, 4, 4)$$

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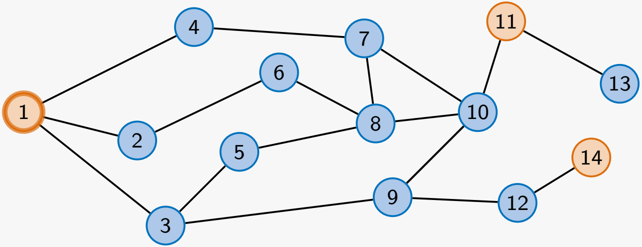

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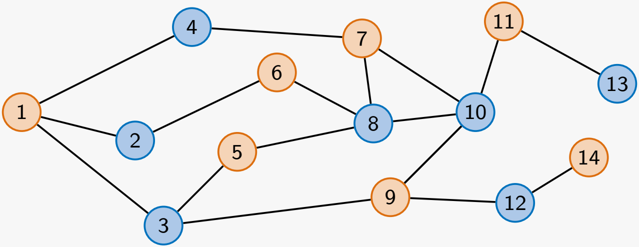

Step 1: $i = 1$, $S_1 = \{1, 5, 6, 7, 9, 11, 14\}$, $\rho_1 = 1$

Create labels $L^1(u)$ for all $u \in S_1$ consisting of *three* fields containing

- the label $L^0(dom_{S_0}(u))$,
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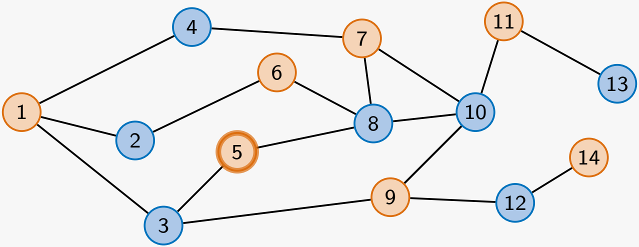

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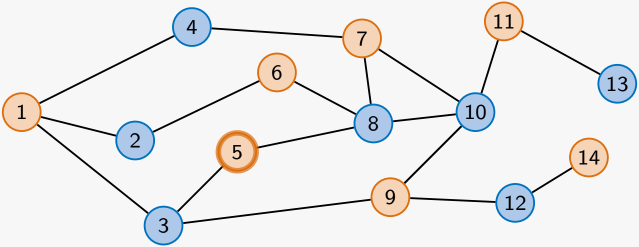

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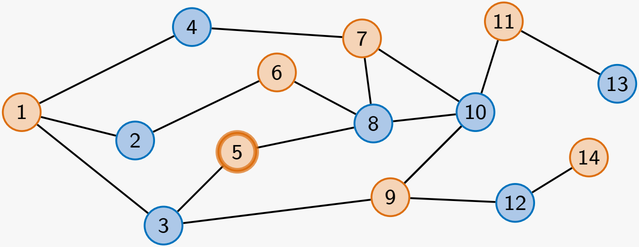

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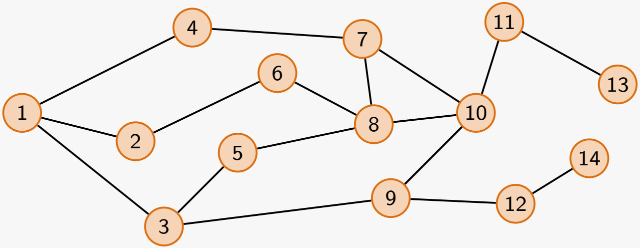

Step 2: $i = 2$, $S_2 = V$, $\rho_2 = 0$

Create labels $L^2(u)$ for all $u \in S_2$ consisting of *three* fields containing

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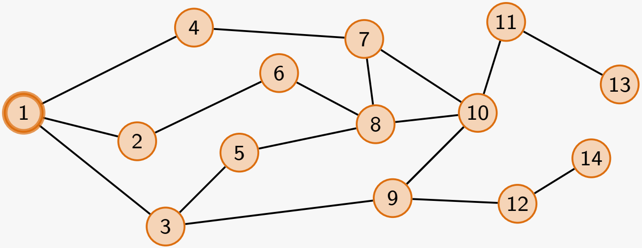

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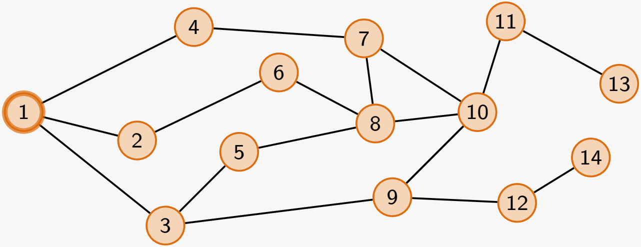

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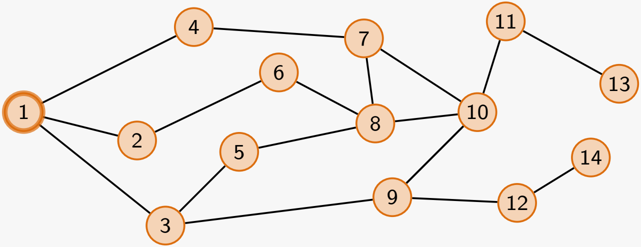

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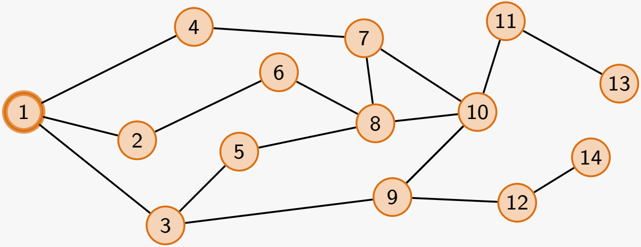
Building the labels



Let's explicitly construct the labels of vertices 1 and 13 for later use:

$$L^2(1) = L^1(\mathrm{dom}_{S_1}(1)) \circ (0, 0, 1, 1, 1, 2, 2, 2, 3, 2, 3, 4, 3, 0, 4)$$

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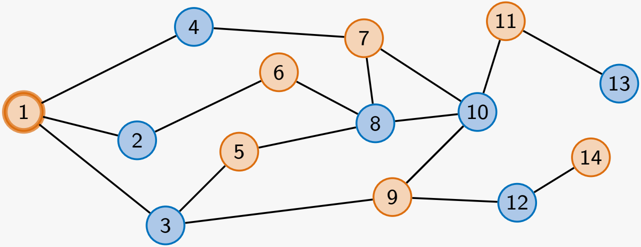

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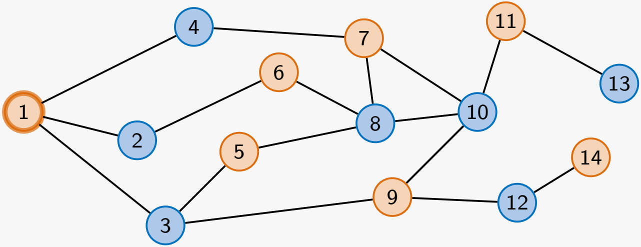

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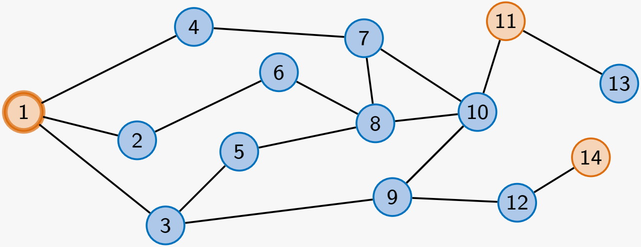

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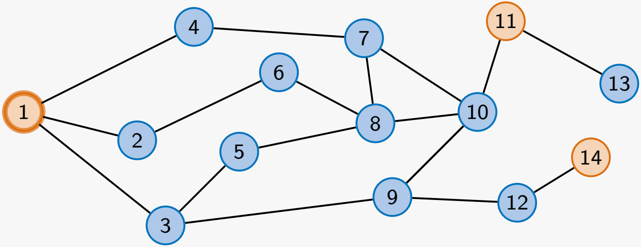

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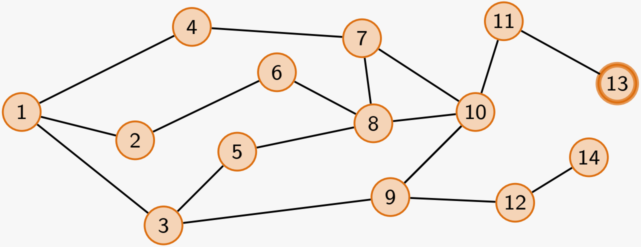

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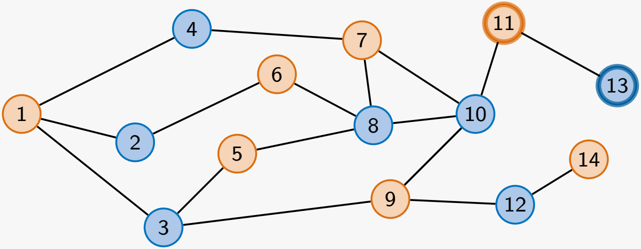
Building the labels



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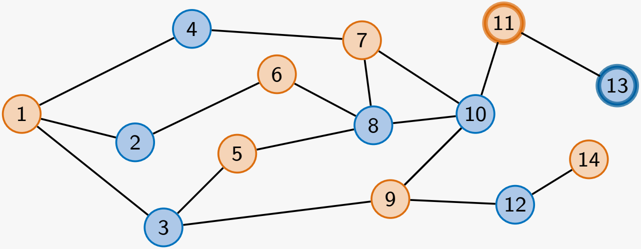

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$$
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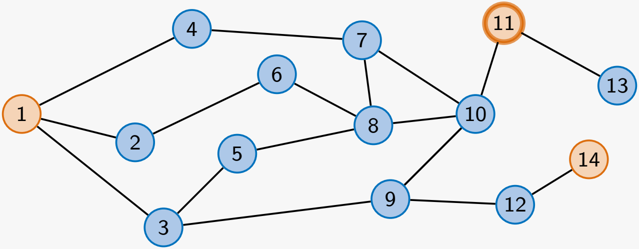
Building the labels



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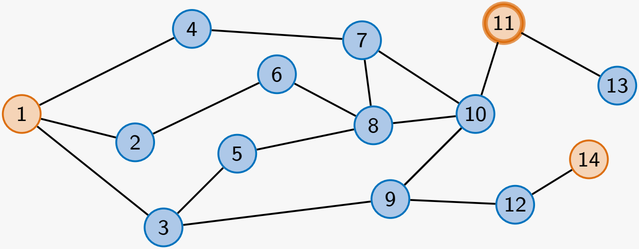

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Recall: $\rho_i = 2^{k-i} - 1,\ |S_i| \leq \frac{n}{2^{k-i}},\ k = \lceil \log \log n \rceil$

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$$\sum_{i=1}^{k} |S_i| \log(4\rho_{i-1} + 1) = \ldots \text{see slide no. } 33 \ldots \leq 8n$$

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$$L_{\max} \leq 9n + \mathcal{O}(\log \log n \log n)$$

Time bounds to create labels

- The dominating collection $\mathcal{S}$ can be determined in $\mathcal{O}(n^2)$, since
  - once the BFS is run which takes $\mathcal{O}(n^2)$
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Labels can be created in $\mathcal{O}(n^2)$

Decoding labels

Idea:

- The distance $d(x, y)$ for $x, y \in S_i$ with $i > 0$ can be calculated using the Lemma about distances between dominators once these are known.
- The distance $d(x', y')$ between dominators of $x', y' \in S_{i-1}$ of x, y can be determined recursively.
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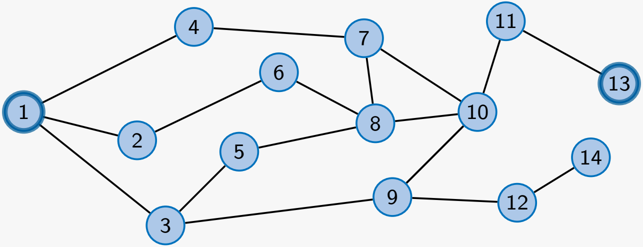
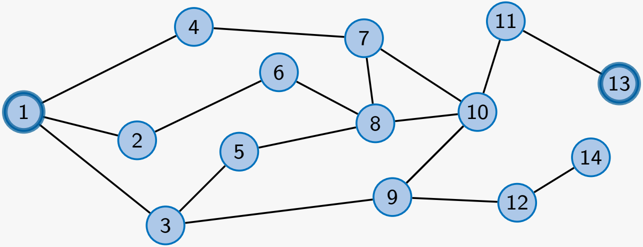
Let's do it by example step by step...

Decoding labels



What's the distance between $x = 1$ **and** $y = 13$**?**

Decoding labels



What's the distance between $x = 1$ and $y = 13$?

Obtain the labels of x, y and determine their dominators in S_1

$$L^2(1) = (0,0,4,4) \circ (0,0,2,2,2,2,4,4) \circ (0,0,1,1,1,2,2,2,3,2,3,4,3,0,4)$$
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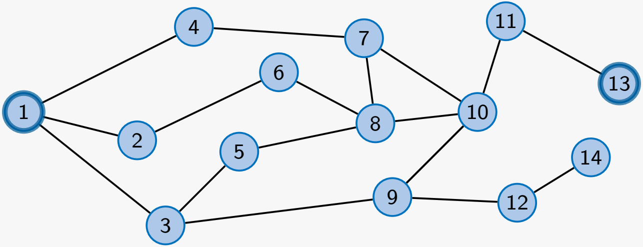

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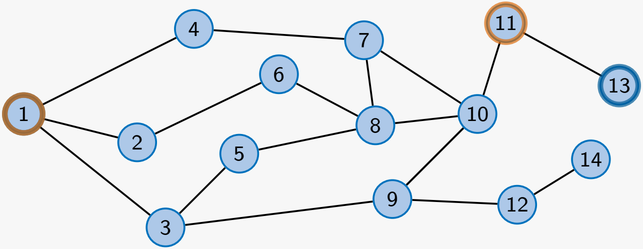

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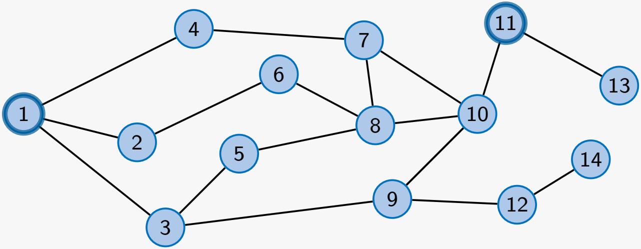
$$L^2(13) = (1, 4, 0, 4) \circ (5, 4, 3, 3, 2, 2, 0, 4) \circ (12, 0, 0, 4, 4, 4, 4, 3, 3, 3, 2, 1, 4, 0, 0)$$

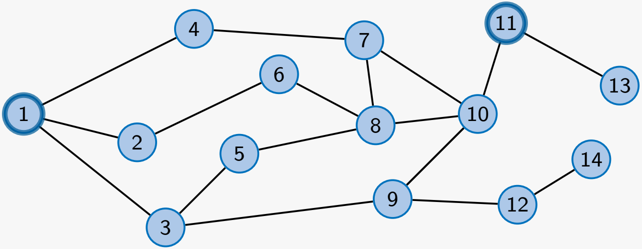
$$\Rightarrow y' = \mathrm{dom}_{S_1}(1) = 1 \text{ and } y' = \mathrm{dom}_{S_1}(13) = 11$$

Decoding labels



Determine $d(x', y') = d(1, 11)$ ← recursive step

Decoding labels



Determine $d(x', y') = d(1, 11)$ ← recursive step

Obtain the labels of x', y' and determine their dominators in S_0

$$L^1(1) = (0,0,4,4) \circ (0,0,2,2,2,2,4,4) \circ (0,0,1,1,1,2,2,2,3,2,3,4,3,0,4)$$
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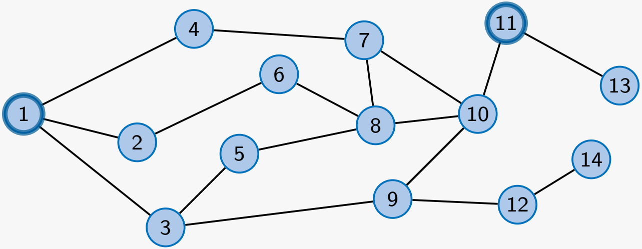

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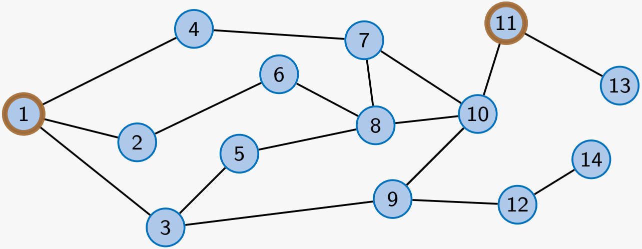

Determine $d(x', y') = d(1, 11) \leftarrow$ recursive step

Obtain the labels of x', y' and determine their dominators in S_0

$$L^1(1) = (0, 0, 4, 4) \circ (0, 0, 2, 2, 2, 2, 4, 4) \circ (0, 0, 1, 1, 1, 2, 2, 2, 3, 2, 3, 4, 3, 0, 4)$$
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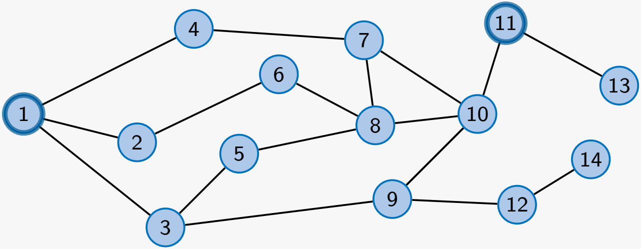
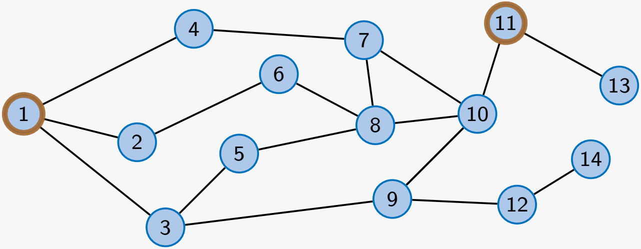
$$\Rightarrow x'' = \mathrm{dom}_{S_0}(x') = 1 \text{ and } y'' = \mathrm{dom}_{S_0}(y') = 11$$

Decoding labels



Determine $d(x'', y'') = d(1, 11)$ ← another recursive step

Decoding labels



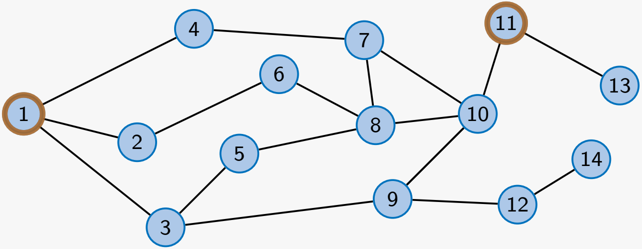
Determine $d(x'', y'') = d(1, 11)$ ← another recursive step

Obtain the labels of x'', y'' and determine $d(x'', y'')$ directly

$$L^0(1) = (0,0,4,4) \circ (0,0,2,2,2,2,4,4) \circ (0,0,1,1,1,2,2,2,3,2,3,4,3,0,4)$$
$$L^0(11) = (1,4,0,4) \circ (5,4,3,3,2,2,0,4) \circ (12,0,0,4,4,4,4,3,3,3,2,1,4,0,0)$$

End of recursion: use rank order(y'') as index in the label of x'' to determine $d(x'', y'')$.

Decoding labels



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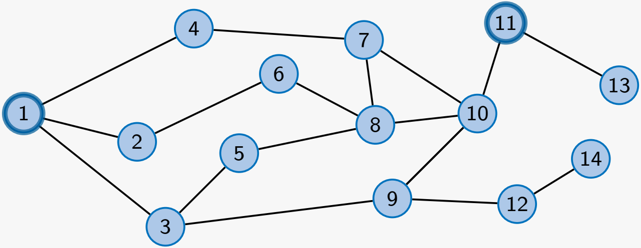
End of recursion: use rank order(y'') as index in the label of x'' to determine $d(x'', y'')$.

$$\Rightarrow d(\mathrm{dom}_{S_0}(x'), \mathrm{dom}_{S_0}(y')) = d(x'', y'') = d(1, 11) = 4$$

Decoding labels



$$L^1(1) = (0,0,4,4) \circ (0,0,2,2,2,2,4,4) \circ (0,0,1,1,1,2,2,2,3,2,3,4,3,0,4)$$
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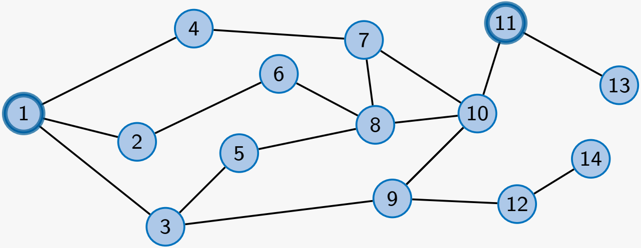
Compute $d(x', y')$ using the lemma about the distance $d(x'', y'')$ between dominators:

$$d(x'', y'') - 2\rho_0 \leq d(x', y') \leq d(x'', y'') + 2\rho_0$$
$$-2 \leq d(x', y') \leq 10.$$

Decoding labels



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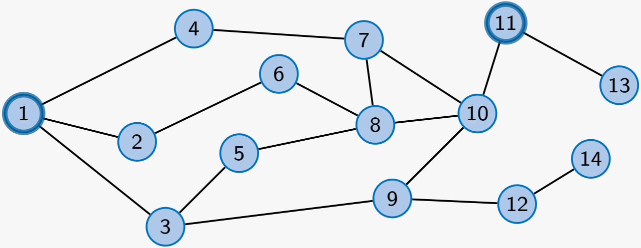

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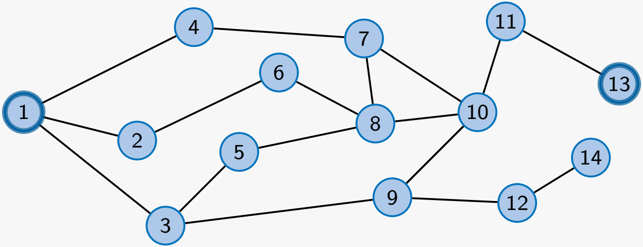
$$\Rightarrow d(\text{dom}_{S_1}(x), \text{dom}_{S_1}(y)) = d(x', y') = 4$$

Decoding labels



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Compute $d(x,y)$ using the lemma about the distance $d(x',y')$ between dominators again:

$$d(x',y') - 2\rho_1 \leq d(x,y) \leq d(x',y') + 2\rho_1$$
$$2 \leq d(x,y) \leq 6.$$

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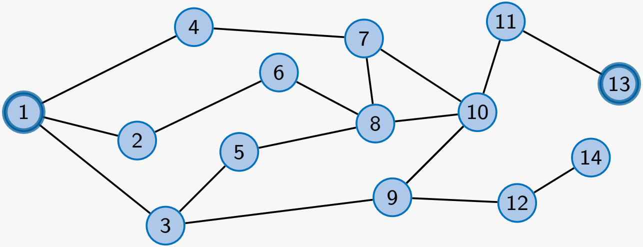

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Furthermore, we know that $d(x,y) \mod (4\rho_1 + 1) = d(x,y) \mod 5 = 0$.

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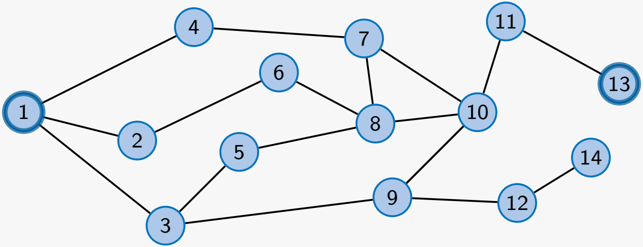

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Furthermore, we know that $d(x,y) \mod (4\rho_1 + 1) = d(x,y) \mod 5 = 0$.

$$\Rightarrow d(x,y) = 5$$

Time needed for decoding

What's the time needed for decoding labels?

- Each step requires to
  - obtain the labels and dominators of x and y,
  - recursive call to determine the distance between dominators,
  - get the rank order(y),
  - and use it as pointer to determine $d(x, y) \mod \rho$.

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$\Rightarrow$ Decoding in $\mathcal{O}(\log \log n)$

Summary

We have seen a distance labeling scheme for general, undirected, and unweighted graphs which allows for

- labels of size $\mathcal{O}(n)$,
- decoding in time $\mathcal{O}(\log\log n)$, and
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Smaller labels are possible for certain classes of graphs, e.g.

- $\Theta(\log^2 n)$ for trees,
- $\mathcal{O}(\sqrt{n}\log n)$ and $\Omega(n^{1/3})$ for planar graphs, and
- $\Omega(\sqrt{n})$ for bounded degree graphs.

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A tradeoff between space and efficiency for routing tables
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David Peleg
Proximity-Preserving Labeling Schemes and Their Applications
ACM, Lecture Notes In Computer Science, vol. 1655, pp. 30–41, 1999.

Recall: $\rho_i = 2^{k-i} - 1,\ |S_i| \leq \frac{n}{2^{k-i}},\ k = \lceil \log \log n \rceil$

$$\begin{aligned}
\sum_{i=1}^{k} |S_i| \log(4\rho_{i-1} + 1) &= \sum_{i=1}^{k} \frac{n}{2^{k-i}} \log(4 \cdot (2^{k-i+1} - 1) + 1) \\
&= n \cdot \sum_{i=1}^{k} \frac{\log(2^{k-i+3} - 3)}{2^{k-i}} \\
&\leq n \cdot \sum_{i=1}^{k} \frac{k - i + 3}{2^{k-i}} \\
&= n \cdot \left(\frac{3}{2^0} + \frac{4}{2^1} + \frac{5}{2^2} + \ldots + \frac{k+2}{2^{k-1}} \right) \\
&= n \cdot \sum_{i=0}^{k-1} \frac{i+3}{2^i}
\end{aligned}$$

$$
\begin{aligned}
= n \cdot \left(\sum_{i=0}^{k-1} \frac{i}{2^i} + 3 \sum_{i=0}^{k-1} \frac{1}{2^i} \right) &= n \cdot \left(\sum_{i=0}^{k-1} \left[\frac{\partial}{\partial x} x^{i+1} \right]_{x=\frac{1}{2}} - \sum_{i=0}^{k-1} \frac{1}{2^i} + 3 \sum_{i=0}^{k-1} \frac{1}{2^i} \right) \\
&= n \cdot \left(\left[\frac{\partial}{\partial x} \sum_{i=0}^{k-1} x^{i+1} \right]_{x=\frac{1}{2}} + 2 \sum_{i=0}^{k-1} \frac{1}{2^i} \right) \\
&\leq n \cdot \left(\left[\frac{\partial}{\partial x} \sum_{i=0}^{\infty} x^{i+1} \right]_{x=\frac{1}{2}} + 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \right) \\
&= n \cdot \left(\left[\frac{\partial}{\partial x} \frac{1}{1-x} \right]_{x=\frac{1}{2}} + \frac{2}{1-\frac{1}{2}} \right) \\
&= n \cdot \left(\left[\frac{1}{(1-x)^2} \right]_{x=\frac{1}{2}} + 4 \right) = 8n
\end{aligned}
$$