

Computational Social Choice

From Arrow's impossibility to Fishburn's maximal lotteries

STACS 2015 Tutorial
Felix Brandt

Motivation

- ▶ *What is “social choice theory”?*
 - ▶ How to aggregate possibly conflicting preferences into collective choices in a fair and satisfactory way?
 - ▶ Origins: mathematics, economics, and political science
 - ▶ Essential ingredients
 - **Autonomous agents** (e.g., human or software agents)
 - A set of **alternatives** (depending on the application, alternatives can be political candidates, resource allocations, coalition structures, etc.)
 - **Preferences** over alternatives
 - **Aggregation functions**
- ▶ The **axiomatic method** will play a crucial role in this tutorial.
 - ▶ Which formal properties should an aggregation function satisfy?
 - ▶ Which of these properties can be satisfied simultaneously?



Handbook of Computational Social Choice

(Cambridge University Press, forthcoming in 2015)

1. Introduction to Computational Social Choice (B., Conitzer, Endriss, Lang, Procaccia)

Part 1: Voting

2. Introduction to the Theory of Voting (Zwicker)
3. Tournament Solutions (B., Brill, Harrenstein)
4. Weighted Tournament Solutions (Fischer, Hudry, Niedermeier)
5. Dodgson's Rule and Young's Rule (Caragiannis, Hemaspaandra, Hemaspaandra)
6. Barriers to Manipulation in Voting (Conitzer, Walsh)
7. Control and Bribery in Voting (Faliszewski, Rothe)
8. Rationalizations of Voting Rules (Elkind, Slinko)
9. Voting in Combinatorial Domains (Lang, Xia)
10. Incomplete Information and Communication in Voting (Boutillier, Rosenschein)

Part 2: Fair Allocation

11. Introduction to the Theory of Fair Allocation (Thomson)
12. Fair Allocation of Indivisible Goods (Bouveret, Chevaleyre, Maudet)
13. Cake Cutting Algorithms (Procaccia)

Part 3: Coalition Formation

14. Matching under Preferences (Klaus, Manlove, Rossi)
15. Hedonic Games (Aziz, Savani)
16. Weighted Voting Games (Chalkiadakis, Wooldridge)

Part 4: Additional Topics

17. Judgment Aggregation (Endriss)
18. The Axiomatic Approach and the Internet (Tennenholtz, Zohar)
19. Knockout Tournaments (Vassilevska-Williams)

Plurality

- ▶ Why are there different voting rules?
 - ▶ What's wrong with **plurality** (the most widespread voting rule) where alternatives that are ranked first by most voters win?
 - ▶ Consider a *preference profile* with 21 voters, who rank four alternatives as in the table on the right.
 - ▶ **Alternative a is the unique plurality winner** despite
 - a majority of voters think a is the worst alternative,
 - a loses against b , c , and d in pairwise majority comparisons, and
 - if the preferences of all voters are reversed, a still wins.
 - ▶ In July 2010, 22 experts on social choice theory met in France and voted on which voting rules should be used. Plurality received *no support* at all (among 18 rules).

3	5	7	6
a	a	b	c
b	c	d	b
c	b	c	d
d	d	a	a



5 Common Voting Rules

- ▶ **Plurality**
 - ▶ Used in most democratic countries, ubiquitous
 - ▶ Alternatives that are ranked first by most voters
- ▶ **Borda**
 - ▶ Used in Slovenia, academic institutions, Eurovision song contest
 - ▶ The most preferred alternative of each voter gets $m-1$ points, the second most-preferred $m-2$ points, etc. Alternatives with highest accumulated score win.
- ▶ **Plurality with runoff**
 - ▶ Used to elect the President of France
 - ▶ The two alternatives that are ranked first by most voters face off in a majority runoff.



5 Common Voting Rules (ctd.)

- ▶ **Instant-runoff**
 - ▶ Used in Australia, Ireland, Malta, Academy awards
 - ▶ Alternatives that are ranked first by the lowest number of voters are deleted. Repeat until no more alternatives can be deleted. The remaining alternatives win.
 - ▶ In the UK 2011 alternative vote referendum, people chose plurality over instant-runoff.
- ▶ **Sequential majority comparisons**
 - ▶ Used by US congress to pass laws (aka *amendment procedure*) and in many committees
 - ▶ Alternatives that win a fixed sequence of pairwise comparisons (e.g., $((a \text{ vs. } b) \text{ vs. } c)$, etc.).



A Curious Preference Profile

33%	16%	3%	8%	18%	22%
<i>a</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>d</i>	<i>e</i>
<i>b</i>	<i>d</i>	<i>d</i>	<i>e</i>	<i>e</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>b</i>
<i>d</i>	<i>e</i>	<i>a</i>	<i>d</i>	<i>b</i>	<i>d</i>
<i>e</i>	<i>a</i>	<i>e</i>	<i>a</i>	<i>a</i>	<i>a</i>

Example due to Michel Balinski

- ▶ Plurality: **a wins**
- ▶ Borda: **b wins**
- ▶ Sequential majority comparisons (any order): **c wins**
- ▶ Instant-runoff: **d wins**
- ▶ Plurality with runoff: **e wins**

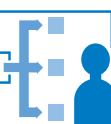


Rational Choice Theory

- ▶ A prerequisite for analyzing **collective choice** is to understand **individual choice**.
- ▶ Let U be a finite **universe of alternatives**.
- ▶ A **choice function** f maps a feasible set $A \subseteq U$ to a choice set $f(A) \subseteq A$.
 - ▶ We require that $f(A) = \emptyset$ only if $A = \emptyset$.
- ▶ Not every choice function complies with our intuitive understanding of rationality.
 - ▶ Certain patterns of choice from varying feasible sets may be deemed inconsistent, e.g., choosing a from $\{a, b, c\}$, but b from $\{a, b\}$.



A	f(A)
ab	a
bc	b
ac	a
abc	a



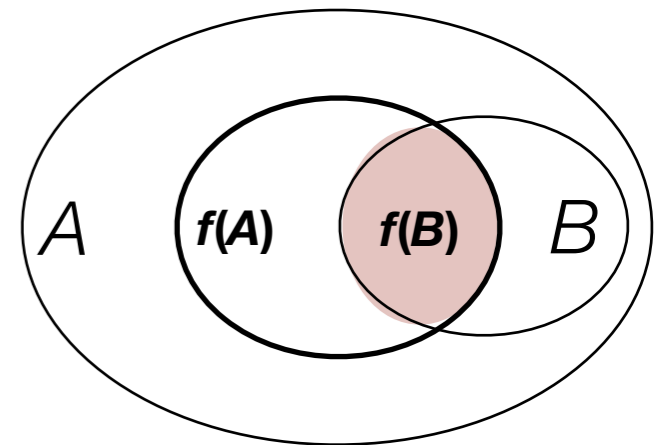
Rationalizable Choice

- ▶ Binary preference relation \succsim on U
 - ▶ $x \succsim y$ is interpreted as “ x is at least as good as y ”.
 - ▶ \succsim is assumed to be transitive and complete.
- ▶ Best alternatives
 - ▶ For a binary relation \succsim and a feasible set A ,
 $Max(\succsim, A) = \{x \in A \mid \nexists y \in A \text{ such that } y \succ x\}$
- ▶ f is **rationalizable** if there exists a preference relation \succsim on U such that $f(A) = Max(\succsim, A)$ for all A .
 - ▶ The previously mentioned choice function f with $f(\{a, b, c\}) = \{a\}$ and $f(\{a, b\}) = \{b\}$ cannot be rationalized.



Consistent Choice

- ▶ It would be a nice if the non-existence of a rationalizing relation could be pointed out by finding **inconsistencies**.
- ▶ f satisfies **consistency** if for all A, B with $B \subseteq A$, $f(A) \cap B \neq \emptyset$ implies $f(B) = f(A) \cap B$.



- ▶ Consequence: If x is chosen from a feasible set, then it is also chosen from all subsets that contain x .
- ▶ Example: Plurality does not satisfy consistency (when scores are computed for each feasible set).
 - $f(\{a, b, c\}) = \{a\}$ and $f(\{a, b\}) = \{b\}$
- ▶ Theorem (Samuelson, 1938; Arrow, 1959): A choice function is rationalizable iff it satisfies consistency.

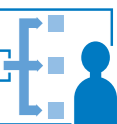
3	2	2
a	b	c
b	c	b
c	a	a



From Choice to Social Choice



- ▶ N is a finite set of at least two voters.
- ▶ $R(U)$ is the set of all preference relations over U .
- ▶ Every $R = (\succsim_1, \dots, \succsim_{|N|}) \in R(U)^{|N|}$ is called a *preference profile*.
- ▶ A *social choice function (SCF)* is a function f that assigns a choice function to each preference profile.
 - ▶ An SCF is rationalizable (consistent) if its underlying choice functions are rationalizable (consistent) for all preference profiles.
 - ▶ We will write $f(R, A)$ as a function of both R and A .
- ▶ Let $n_{xy} = |\{i \in N \mid x \succsim_i y\}|$ and define the *majority rule relation* as $(x R_M y) \Leftrightarrow n_{xy} > n_{yx}$.



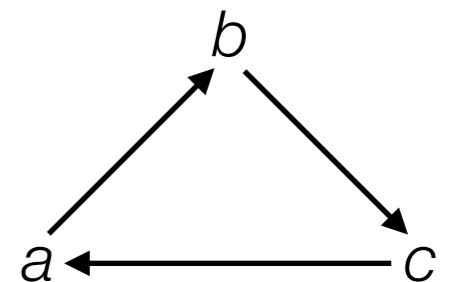
Condorcet's Paradox



Marquis de Condorcet

- ▶ Social choice from **feasible sets of size two** is easy.
 - ▶ The **majority rule SCF** is defined as $f(R, \{x, y\}) = \text{Max}(R_M, \{x, y\})$.
 - ▶ Majority rule can easily be characterized using uncontroversial axioms (e.g., May, 1952).
- ▶ Problems arise whenever there are **more than two alternatives**.
 - ▶ Condorcet paradox (1785): R_M can be intransitive.
 - ▶ Alternative x is a **Condorcet winner** in A if $x R_M y$ for all $y \in A \setminus \{x\}$.
 - ▶ An SCF f is a **Condorcet extension** if $f(R, A) = \{x\}$ whenever x is a Condorcet winner in A .

1	1	1
a	b	c
b	c	a
c	a	b



Arrow's Impossibility



Kenneth J. Arrow

- ▶ An SCF satisfies *independence of infeasible alternatives (IIA)* if the choice set only depends on preferences over alternatives within the feasible set.
- ▶ An SCF satisfies *Pareto-optimality* if an alternative will not be chosen if there exists another alternative such that all voters prefer the latter to the former.
- ▶ An SCF is *dictatorial* if there exists a voter whose most preferred alternative is always uniquely chosen.
- ▶ Theorem (Arrow, 1951): *Every rationalizable SCF that satisfies IIA and Pareto-optimality is dictatorial* when $|U| \geq 3$.
 - ▶ Nipkow (2009) has verified a proof of Arrow's theorem using Isabelle.
 - ▶ Tang & Lin (2009) reduced the statement to a finite base case that was solved by a computer.



What now?

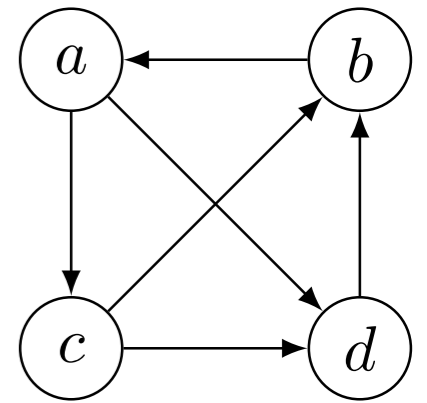
- ▶ **Rationalizability** (or, equivalently, **consistency**) is incompatible with collective choice when $|U| \geq 3$.
 - ▶ Dropping IIA offers little relief (Banks, 1995).
 - ▶ Dropping Pareto-optimality offers little relief (Wilson, 1972).
 - ▶ Dropping non-dictatorship is unacceptable.
- ▶ In this tutorial, we will consider two escape routes from Arrow's impossibility:
 - ▶ SCFs that satisfy **weaker notions of consistency**
 - Top cycle, uncovered set, Banks set, tournament equilibrium set
 - ▶ **Randomized SCFs**
 - Random dictatorship, maximal lotteries



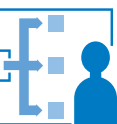
Weakly Consistent SCFs

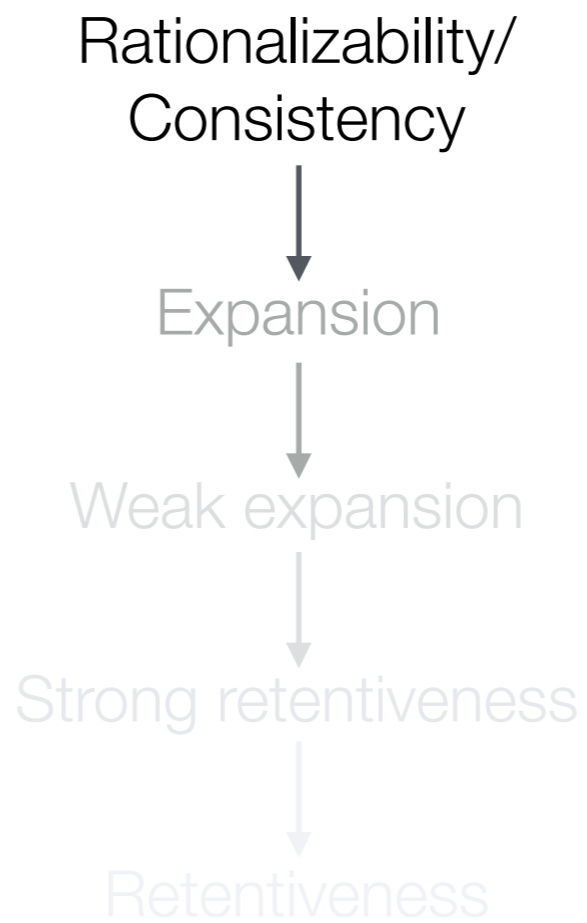
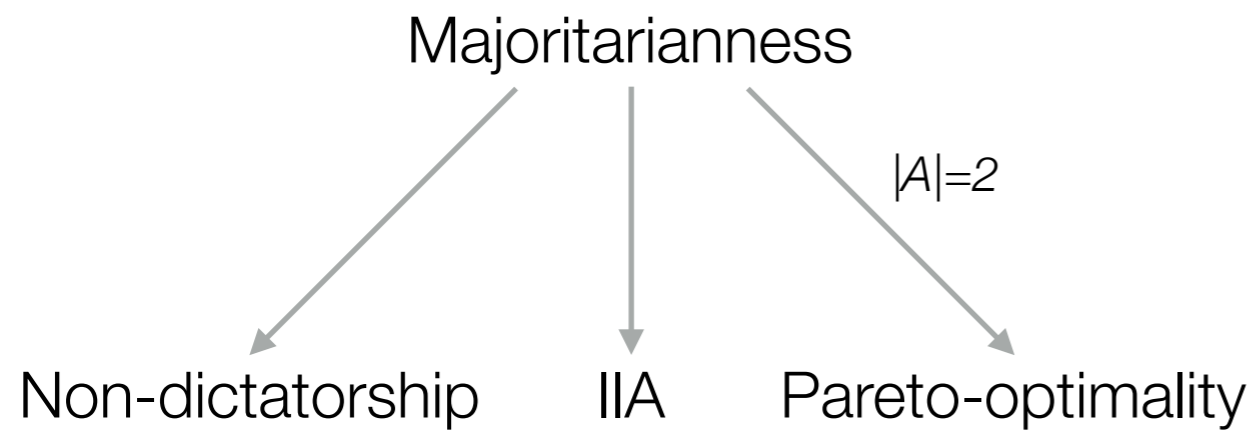


Tournaments

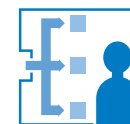


- ▶ For a given preference profile R , a feasible set A and majority rule R_M define a **directed graph** (A, R_M) .
 - ▶ We say that b **dominates** a if $b R_M a$.
 - ▶ Every asymmetric directed graph is induced by some preference profile (McGarvey, 1953).
- ▶ A **majoritarian SCF** is an SCF whose output only depends on (A, R_M) .
 - ▶ For simplicity, we will assume that individual preferences are anti-symmetric and that $|N|$ is odd. Hence, (A, R_M) is a **tournament**.
 - ▶ SCF f is said to be **finer** than SCF g if $f \subseteq g$.
 - ▶ **Dominion** $D(x) = \{y \in A \mid x R_M y\}$
 - ▶ **Dominators** $\bar{D}(x) = \{y \in A \mid y R_M x\}$





Arrow's Impossibility



The Top Cycle

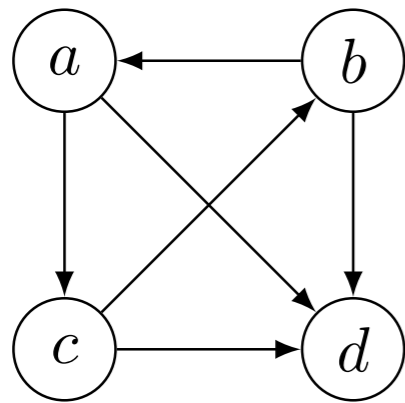


John I. Good

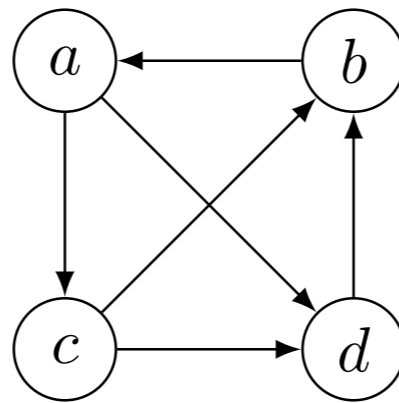
- ▶ Consistency can be weakened to **expansion**:
 $B \subseteq A$ and $f(A) \cap B \neq \emptyset$ implies $f(B) \subseteq f(A)$.
- ▶ Theorem (Bordes, 1976): There is a unique finest majoritarian SCF satisfying **expansion**: the top cycle.
- ▶ A **dominant set** is a nonempty set of alternatives $B \subseteq A$ such that for all $x \in B$ and $y \in A \setminus B$, $x R_M y$.
 - ▶ The set of dominant sets is totally ordered by set inclusion (Good, 1971).
 - ▶ Hence, every tournament contains a unique minimal dominant set called the **top cycle (TC)**.
 - ▶ **TC** is a **Condorcet extension**.



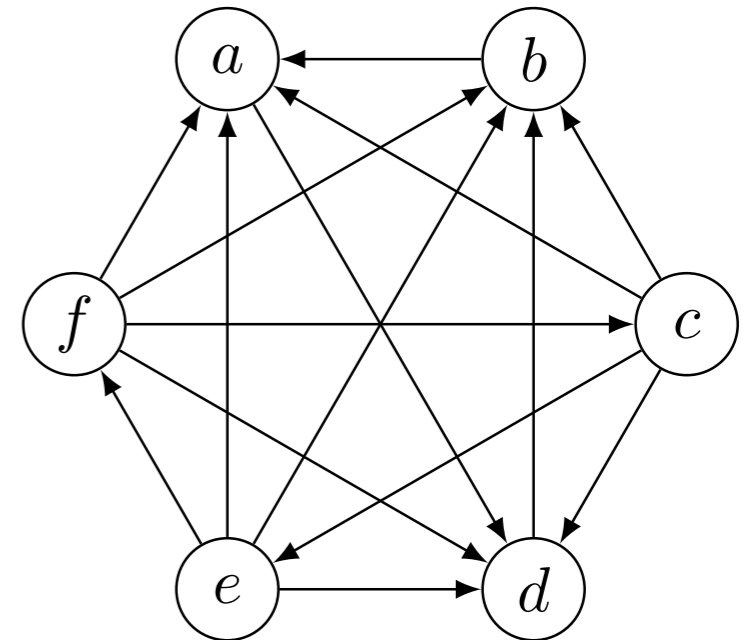
Examples



$$TC(A, R_M) = \{a, b, c\}$$



$$TC(A, R_M) = \{a, b, c, d\}$$



$$TC(A, R_M) = \{c, e, f\}$$



Transitive Closure

- ▶ The essence of Condorcet's paradox and Arrow's impossibility is that majority rule fails to be **transitive**.
 - ▶ Why not just take the transitive (reflexive) closure R_M^* ?
- ▶ Theorem (Deb, 1977): $TC(A, R_M) = Max(R_M^*, A)$.
- ▶ Consequences
 - ▶ TC itself is a cycle. It is the source component in the DAG (directed acyclic graph) of strongly connected components.
 - ▶ Linear-time algorithms for computing TC using Kosaraju's or Tarjan's algorithm for finding strongly connected components
 - Alternatively, one can initialize working set B with all alternatives of maximal outdegree and then iteratively add all alternatives that dominate an alternative in B until no more such alternatives can be found.



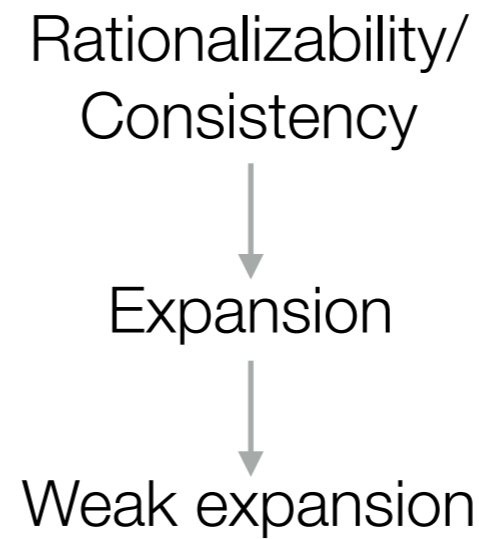
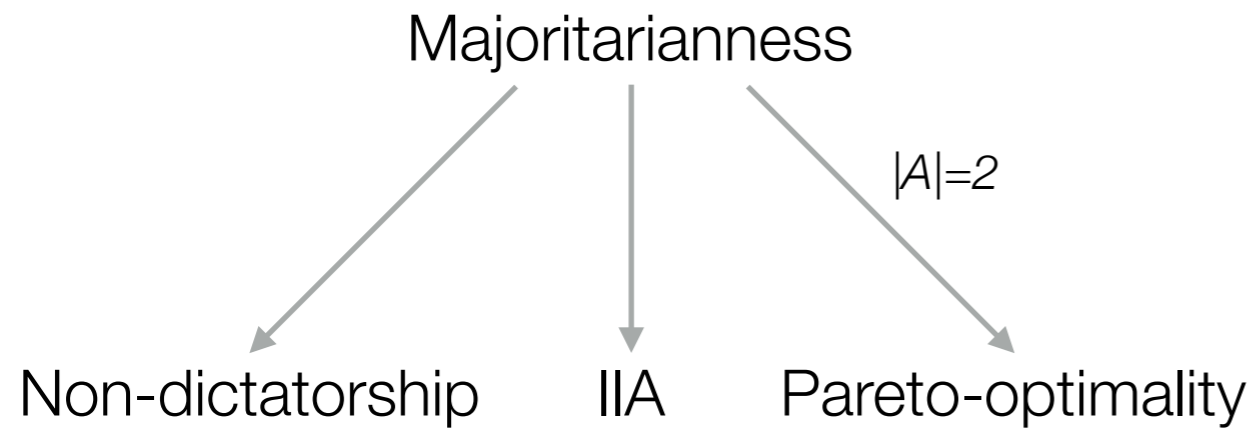
Top Cycle and Pareto-Optimality

- ▶ The top cycle is very large.
- ▶ In fact, it is so large that it **fails to be Pareto-optimal** when there are more than three alternatives (Ferejohn & Grether, 1977).

1	1	1
<i>a</i>	<i>b</i>	<i>d</i>
<i>b</i>	<i>c</i>	<i>a</i>
<i>c</i>	<i>d</i>	<i>b</i>
<i>d</i>	<i>a</i>	<i>c</i>

- ▶ Since Pareto-optimality is an essential ingredient of Arrow's impossibility, this escape route is (so far) not entirely convincing.
 - ▶ Although, technically, Arrow's theorem only requires Pareto-optimality for two-element sets (which the top cycle satisfies).





Arrow's Impossibility

Top Cycle (TC)





Peter C. Fishburn

The Uncovered Set

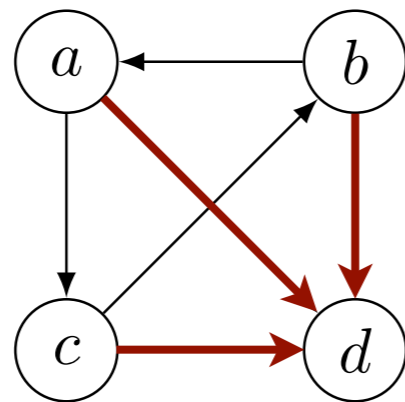


Nicholas Miller

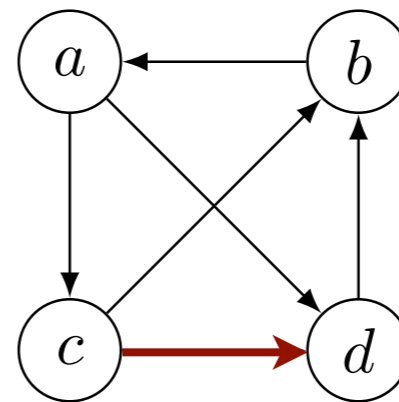
- ▶ Expansion can be further weakened to **weak expansion**:
 $f(A) \cap f(B) \subseteq f(A \cup B)$.
- ▶ Theorem (Moulin, 1986): There is a unique finest majoritarian SCF satisfying **weak expansion**: the uncovered set.
- ▶ Given a tournament (A, R_M) , **x covers y** ($x C y$), if $D(y) \subset D(x)$.
 - ▶ Proposed independently by Fishburn (1977) and Miller (1980)
 - ▶ Transitive subrelation of majority rule
 - ▶ The **uncovered set (UC)** consists of all uncovered alternatives, i.e.,
 $UC(A, P_M) = Max(C, A)$.



Examples



$$UC(A, R_M) = \{a, b, c\}$$



$$UC(A, R_M) = \{a, b, c\}$$

$$TC(A, R_M) = \{a, b, c, d\}$$



Properties of the Uncovered Set

- ▶ Since expansion \Rightarrow weak expansion, $UC \subseteq TC$.
 - ▶ UC is a Condorcet extension.
- ▶ UC satisfies Pareto-optimality.
 - ▶ Theorem (B. and Geist, 2014): UC is the largest majoritarian SCF satisfying Pareto-optimality.
- ▶ How can the uncovered set be efficiently computed?
 - ▶ Straightforward $O(n^3)$ algorithm that computes the covering relation for every pair of alternatives
 - ▶ Can we do better than that?

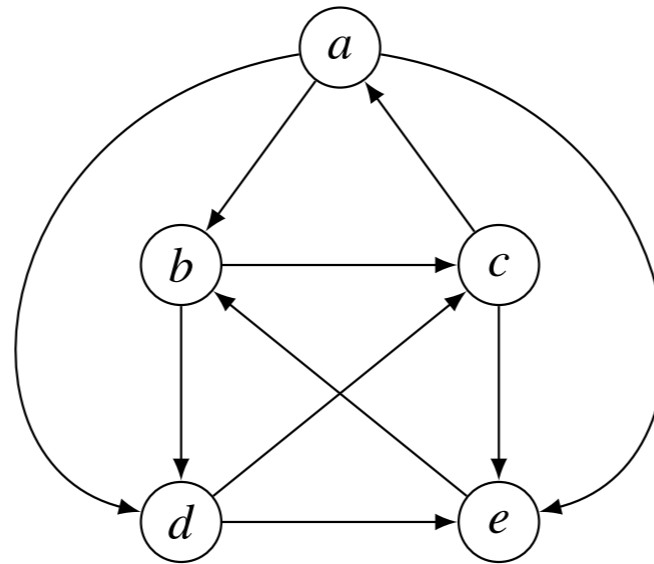


Uncovered Set Algorithm

- ▶ Equivalent characterization of UC
 - ▶ Theorem (Shepsle & Weingast, 1984): UC consists precisely of all alternatives that reach every other alternative in **at most two steps**.
 - Such alternatives are called **kings** in graph theory.
- ▶ Hence, UC can be computed by squaring the tournament's adjacency matrix.
 - ▶ Fastest known **matrix multiplication** algorithm (Le Gall, 2014): $O(n^{2.3728639})$
 - ▶ Just slightly faster than Vassilevska Williams, 2011: $O(n^{2.372873})$
 - ▶ Based on Coppersmith & Winograd (1990): $O(n^{2.376})$
 - ▶ Matrix multiplication is believed to be feasible in linear time ($O(n^2)$).

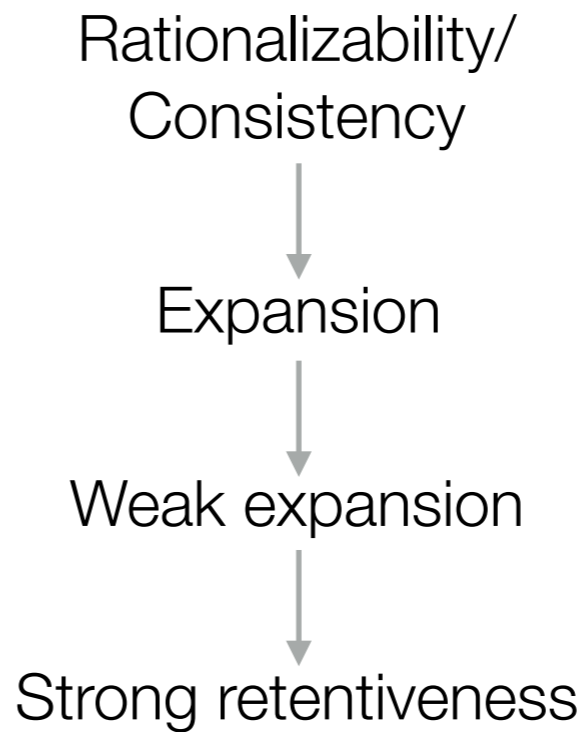
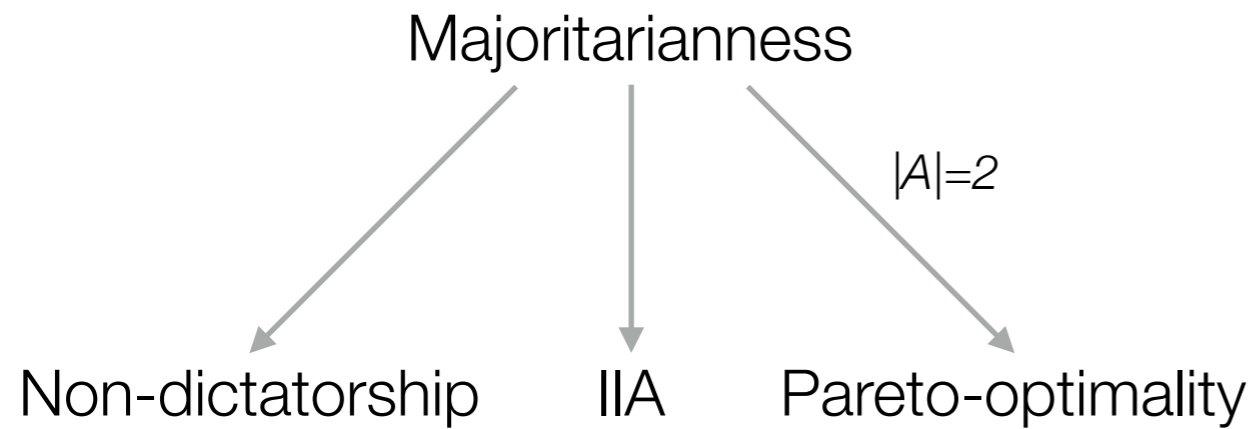


Uncovered Set Algorithm (Example)



$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \color{red}{(0)} & 1 & 1 & 1 & 1 \end{pmatrix}$$





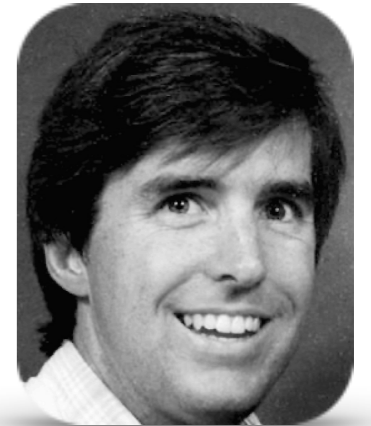
Arrow's Impossibility

Top Cycle (TC)

Uncovered Set (UC)

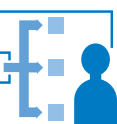


Banks Set



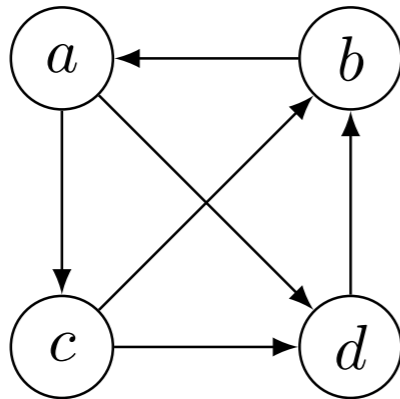
Jeffrey S. Banks

- ▶ Weak expansion can be weakened to **strong retentiveness**: $f(\bar{D}(x)) \subseteq f(A)$ for all $x \in A$.
- ▶ Theorem (B., 2011): There is a unique finest majoritarian SCF satisfying **strong retentiveness**: the Banks set.
- ▶ A **transitive subset** of a tournament (A, R_M) is a set of alternatives $B \subseteq A$ such that R_M is transitive within B .
- ▶ Let $Trans(A, R_M) = \{B \subseteq A \mid B \text{ is transitive}\}$.
- ▶ The **Banks set** (BA) consists of the maximal elements of all inclusion-maximal transitive subsets (Banks, 1985), i.e., $BA(A, R_M) = \{Max(R_M, B) \mid B \in Max(\supseteq, Trans(A, R_M))\}$



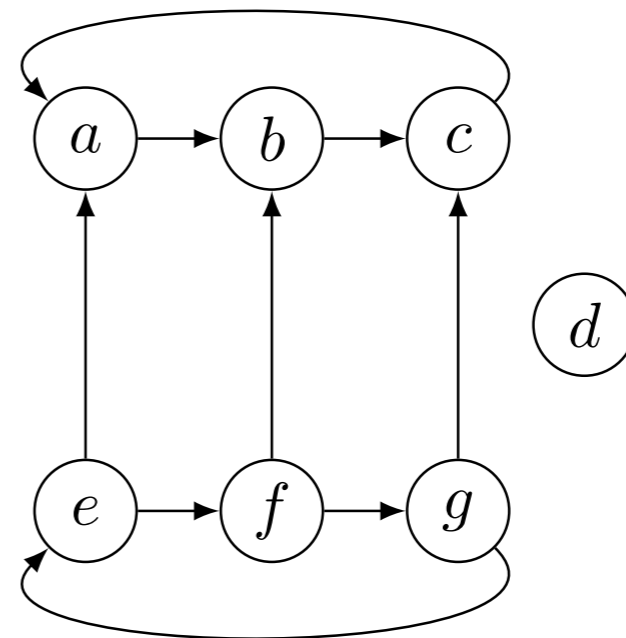
Examples

(All missing edges are pointing downwards.)



$$UC(A, R_M) = \{a, b, c\}$$

$$BA(A, R_M) = \{a, b, c\}$$



$$TC(A, R_M) = \{a, b, c, d, e, f, g\}$$

$$UC(A, R_M) = \{a, b, c, d\}$$

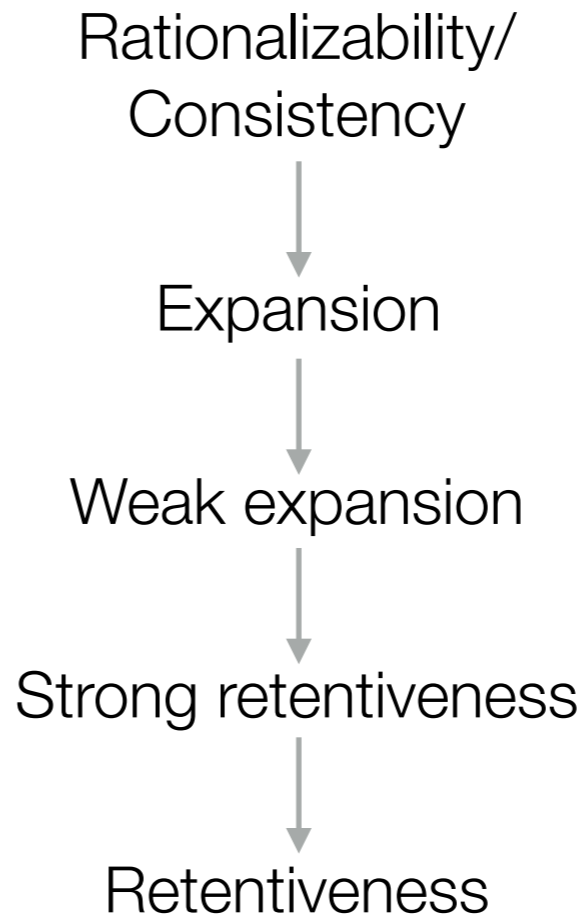
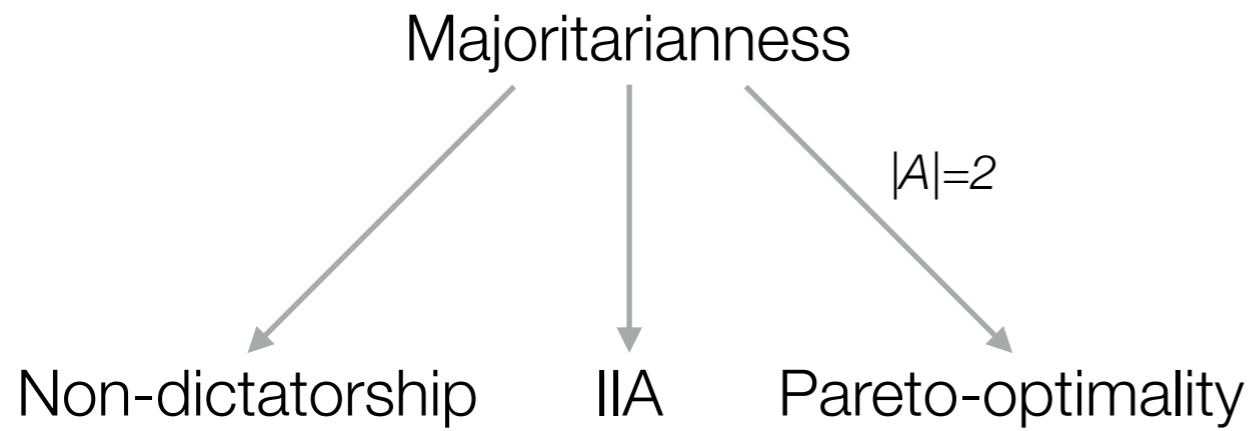
$$BA(A, R_M) = \{a, b, c\}$$



Properties of the Banks Set

- ▶ Since expansion \Rightarrow weak expansion \Rightarrow strong retentiveness,
 $BA \subseteq UC \subseteq TC$.
 - ▶ As a consequence, BA is a **Condorcet extension** and satisfies **Pareto-optimality**.
- ▶ **Random alternatives** in BA can be found in linear time by iteratively constructing maximal transitive sets.
- ▶ Yet, computing the Banks set is **NP-hard** (Woeginger, 2003) and remains NP-hard even for 5 voters (B. et al., 2013).
- ▶ Strong retentiveness can be further weakened to **retentiveness**:
 $f(\bar{D}(x)) \subseteq f(A)$ for all $x \in f(A)$.





Arrow's Impossibility

Top Cycle (TC)

Uncovered Set (UC)

Banks Set (BA)

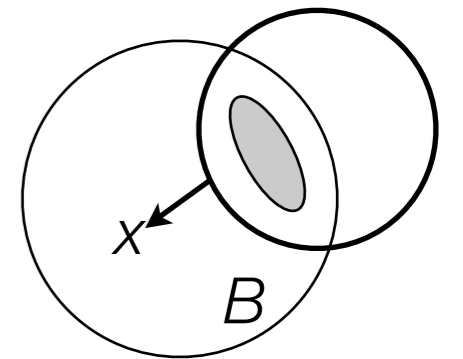


Tournament Equilibrium Set



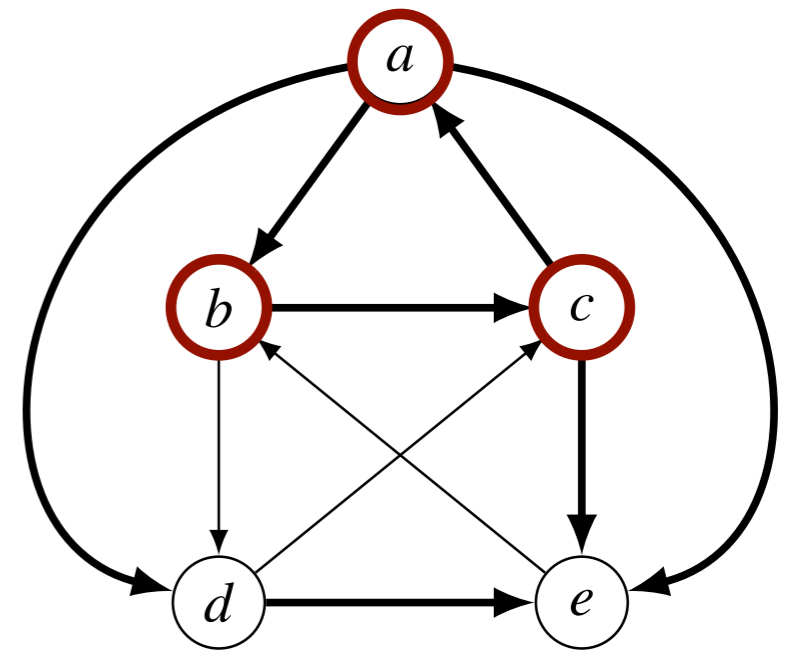
Thomas Schwartz

- ▶ Let f be an arbitrary choice function.
 - ▶ A non-empty set of alternatives B is *f -retentive* if $f(\bar{D}(x)) \subseteq B$ for all $x \in B$.
 - ▶ Idea: No alternative in the set should be “properly” dominated by an outside alternative.
- ▶ $\overset{\circ}{f}$ is a new choice function that yields the **union of all inclusion-minimal f -retentive sets**.
 - ▶ $\overset{\circ}{f}$ satisfies retentiveness.
- ▶ The tournament equilibrium set (TEQ) of a tournament is defined as $TEQ = T\overset{\circ}{E}Q$.
 - ▶ Recursive definition (unique fixed point of ring-operator)
 - ▶ Theorem (Schwartz, 1990): $TEQ \subseteq BA$.

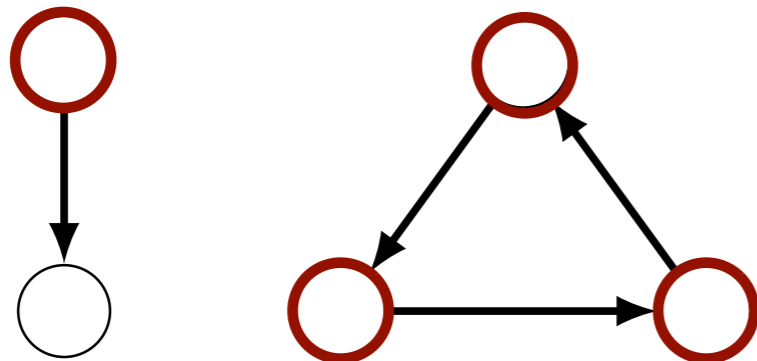


Example

- ▶ $\{a, b, c\}$ is the unique minimal *TEQ*-retentive set.
 - ▶ $TEQ(\bar{D}(a)) = TEQ(\{c\}) = \{c\}$
 - ▶ $TEQ(\bar{D}(b)) = TEQ(\{a, e\}) = \{a\}$
 - ▶ $TEQ(\bar{D}(c)) = TEQ(\{b, d\}) = \{b\}$
 - ▶ $TEQ(\bar{D}(d)) = TEQ(\{a, b\}) = \{a\}$
 - ▶ $TEQ(\bar{D}(e)) = TEQ(\{a, c, d\}) = \{a, c, d\}$



A thick edge from y to x denotes that $y \in TEQ(\bar{D}(x))$.

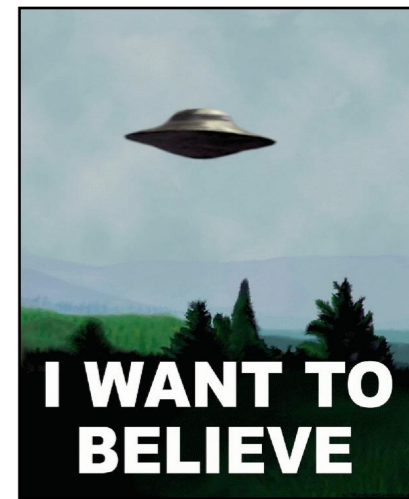


Properties of TEQ

- ▶ Computing *TEQ* is NP-hard (B. et al., 2010) and remains NP-hard even for 7 voters (Bachmeier et al., 2015).
 - ▶ The best known upper bound is PSPACE!
- ▶ Theorem (Laffond et al., 1993; Houy 2009; B., 2011; B. and Harrenstein, 2011): The following statements are equivalent:
 - ▶ Every tournament contains a **unique** minimal *TEQ*-retentive set. (Schwartz' Conjecture, 1990)
 - ▶ *TEQ* is the unique finest majoritarian SCF satisfying retentiveness.
 - ▶ *TEQ* satisfies **monotonicity** (and many other desirable properties).
- ▶ All or nothing:
Either *TEQ* is a most appealing SCF or it is severely flawed.



Schwartz's Conjecture



- ▶ There exists **no counterexample with less than 13 alternatives** (154 billion tournaments have been checked).
 - ▶ *TEQ* satisfies all nice properties if $|A| < 13$.
- ▶ **No counterexample** was found by searching **billions of random tournaments** with up to 50 alternatives.
 - ▶ Checking significantly larger tournaments is intractable.
- ▶ Many **non-trivial weakenings** of Schwartz's conjecture are known to hold (Good, 1971; Dutta, 1988; B. et al., 2010; B., 2011).
- ▶ Theorem (B., Chudnovsky, Kim, Liu, Norin, Scott, Seymour, and Thomassé, 2012): **Schwartz's conjecture is false.**



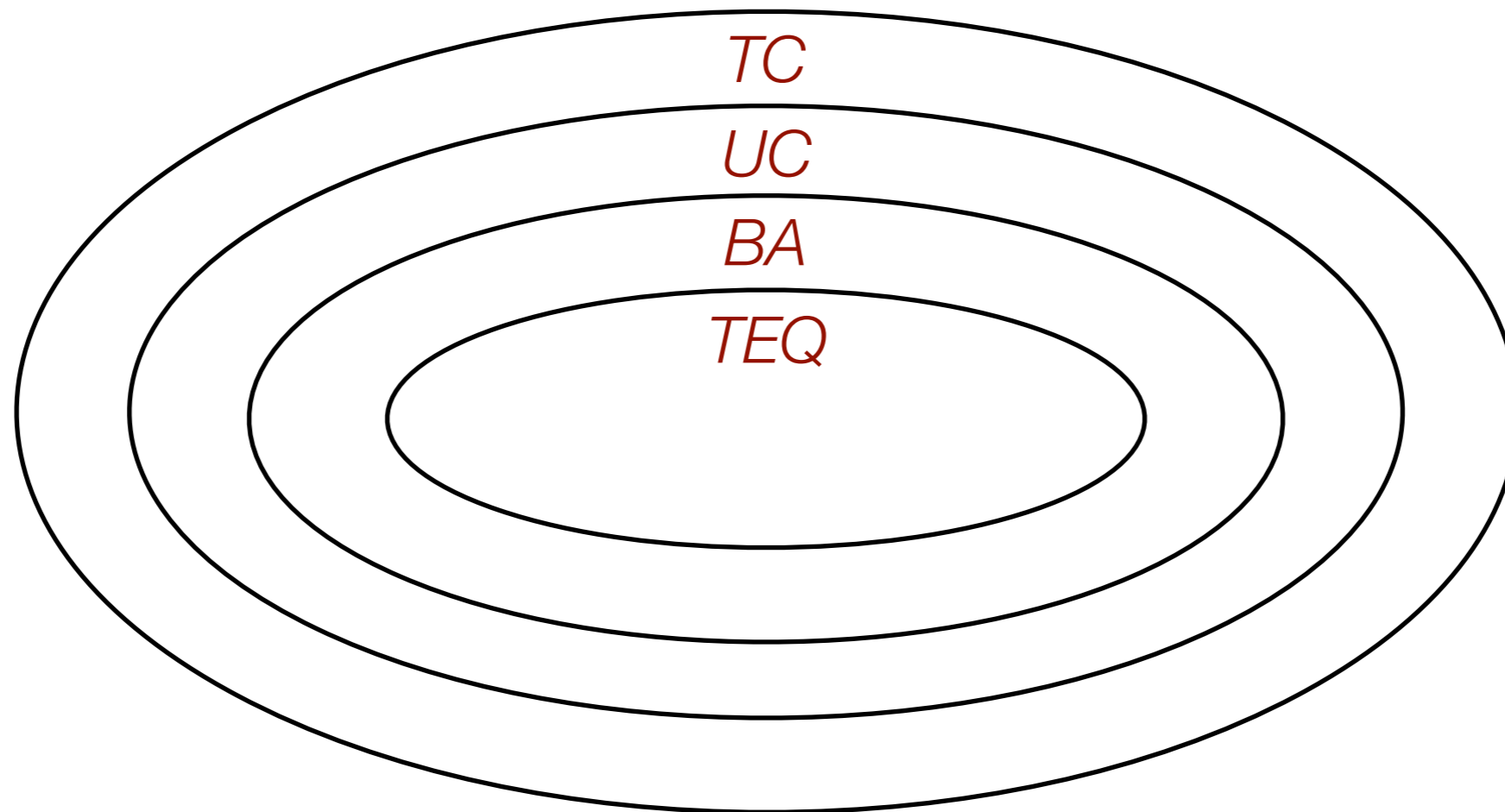
Aftermath



- ▶ Non-constructive proof relying on a probabilistic argument by Erdős and Moser (1964)
 - ▶ Neither the counter-example nor its size can be deduced from proof.
 - ▶ Smallest counter-example of this type requires **about 10^{136} alternatives**.
- ▶ More recently, a counter-example with 24 alternatives was found with the help of a computer (B. & Seidig, 2013).
- ▶ In principle, *TEQ* is **severely flawed**. However, counter-examples are so extremely rare that this has no practical consequences.
 - ▶ This casts doubt on the axiomatic method.



Weakly Consistent SCFs



Top Cycle (1971)	<i>TC</i>	expansion	$O(n^2)$
Uncovered Set (1977)	<i>UC</i>	weak expansion	$O(n^{2.38})$
Banks Set (1985)	<i>BA</i>	strong retentiveness	$2^{O(n)}$
Tournament Equilibrium Set (1990)	<i>TEQ</i>	(retentiveness)	$2^{O(n)}$



Randomized SCFs



Random Dictatorship



Allan Gibbard

- ▶ A **randomized SCF** maps a preference profile to a lottery (probability distribution) over the alternatives.
- ▶ Perhaps the most notorious randomized SCF is random dictatorship.
 - ▶ **One agent is picked uniformly at random** and his most preferred alternative is implemented as the social choice.
- ▶ Random dictatorship is not as bad as it may sound.
 - ▶ It satisfies most of the axioms that are usually considered in social choice theory.
- ▶ Random dictatorship is the only Pareto-optimal randomized SCF that is **strategyproof**, i.e., it cannot be manipulated by lying about one's preferences (Gibbard, 1977).





Germaine Kreweras

Maximal Lotteries



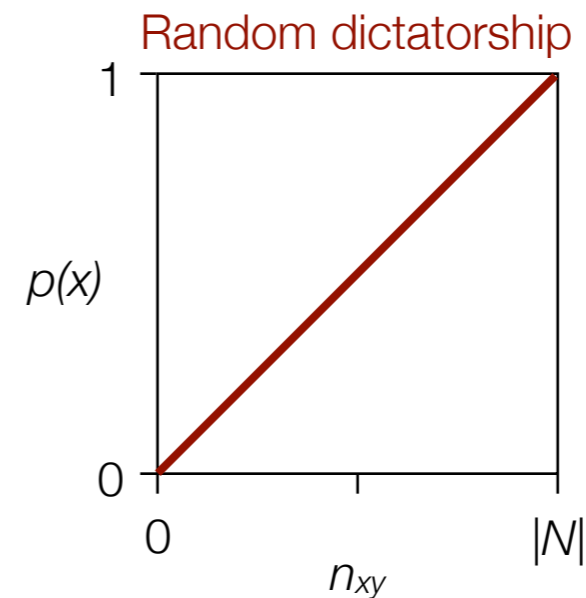
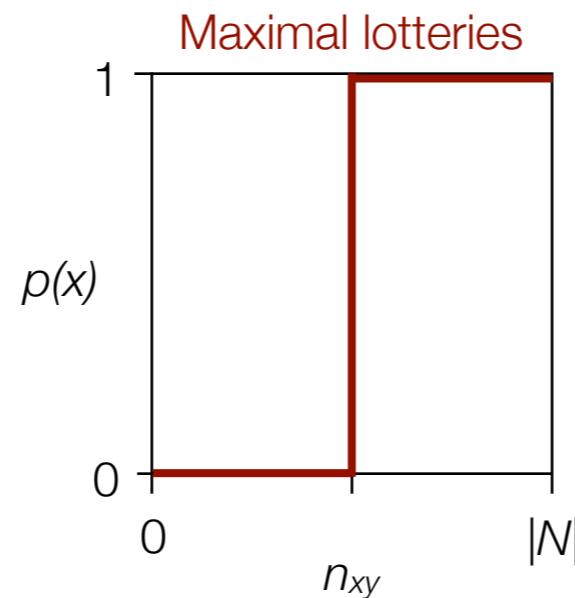
Peter C. Fishburn

- ▶ Kreweras (1965) and Fishburn (1984)
 - ▶ Rediscovered by Laffond et al. (1993), Felsenthal and Machover (1992), Fisher and Ryan (1995), Rivest and Shen (2010)
- ▶ Let $g(x,y) = n_{xy} - n_{yx}$ be the *majority margin* of x and y .
- ▶ Alternative x is a (weak) Condorcet winner if $g(x,y) \geq 0$ for all y .
- ▶ Extend g to lotteries: $g(p,q) = \sum_{x,y} p(x) \cdot q(y) \cdot g(x,y)$
 - ▶ Expected majority margin
- ▶ p is a *maximal lottery* if $g(p,q) \geq 0$ for all q .
 - ▶ *Randomized (weak) Condorcet winner*
 - ▶ Always *exists* due to Minimax Theorem (v. Neumann, 1928)



Examples

- ▶ Two alternatives



- ▶ R can be transformed into a symmetric zero-sum game.
 - ▶ Maximal lotteries are **mixed minimax strategies (or Nash equilibria)**.

2	2	1
a	b	c
b	c	a
c	a	b

	a	b	c
a	0	1	-1
b	-1	0	3
c	1	-3	0

- ▶ The unique maximal lottery is $3/5 a + 1/5 b + 1/5 c$.



Properties of Maximal Lotteries (*ML*)

- ▶ Maximal lotteries are almost always unique.
 - ▶ Always unique for odd number of voters (Laffond et al., 1997)
- ▶ *ML* **does not require** asymmetry, completeness, or even transitivity of preferences.
 - ▶ Random dictatorship requires unique maximum.
 - ▶ Canonical generalization (*RSD*) requires at least one maximum.
- ▶ *ML* can be **efficiently computed** via linear programming.
 - ▶ Computing *RSD* probabilities, on the other hand, is **#P-complete** (Aziz et al., 2013).
- ▶ In the assignment domain, maximal lotteries are known as **popular mixed matchings** (Kavitha et al., 2011).



Properties of Maximal Lotteries (ctd.)

- ▶ **Pareto-dominated** alternatives always get zero probability in every maximal lottery.
 - ▶ In fact, ML is even **efficient** with respect to **stochastic dominance**.
 - ▶ No lottery gives more expected utility for any utility representation consistent with the voters' preferences (Aziz et al., 2012).
 - ▶ Violated by RSD (Bogomolnaia and Moulin, 2001).
- ▶ ML is **weakly strategyproof** in a well-defined sense (Aziz et al., 2013).
- ▶ ML can be uniquely characterized using a version of **consistency** for randomized SCFs (Brandl et al., 2015).



Recommended Literature

- ▶ Books
 - ▶ Allingham: *Choice Theory - A very short introduction*. Oxford University Press, 2002
 - ▶ Austen-Smith and Banks: *Positive Political Theory I*, University of Michigan Press, 1999
 - ▶ Gärtner: *A Primer in Social Choice Theory*, Oxford University Press, 2009
 - ▶ Moulin: *Axioms of Cooperative Decision Making*. Cambridge University Press, 1988
 - ▶ Nitzan: *Collective Choice and Preference*. Cambridge University Press, 2010
- ▶ Introductory book chapter
 - ▶ B., Conitzer, and Endriss. *Computational Social Choice*. In "Multiagent Systems" (G. Weiss, ed.), MIT Press, 2013.

