The Complexity of Constraint Satisfaction Problems

Manuel Bodirsky

Institut für Algebra, TU Dresden

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Outline

1. Constraint Satisfaction Problems: definition and examples

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2. Complexity classification of finite domain CSPs:
   The universal-algebraic approach

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3. The algebraic approach for infinite-domain CSPs

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2. Complexity classification of finite domain CSPs:
   The universal-algebraic approach
3. The algebraic approach for infinite-domain CSPs
4. Complexity of CSPs over the integers, the rationals, and the reals.

Constraint Satisfaction Problems

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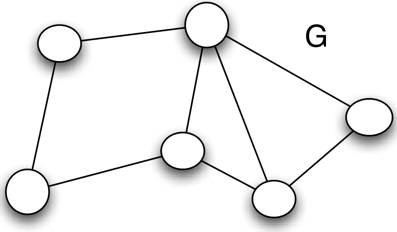
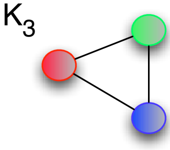
CSP(Γ) is the computational problem to decide whether a given finite τ-structure A homomorphically maps to Γ.

Example: 3-colorability is CSP(K_3)





More Examples of CSPs

Positive 1-in-3-3SAT

Input: A set V and a subset T of V^3.

Question: Is there a map $V \rightarrow \{0,1\}$ such that exactly one entry in each triple in T is mapped to 1?

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Complexity: In P (e.g. by depth-first search)

Logic Formulation of the CSP

Let $\Gamma = (D; R_1, \dots, R_l)$ be a τ-structure.

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Input: A primitive positive τ-sentence Φ,
i.e., a first-order sentence of the form

$$\exists x_1, \ldots, x_n \, (\psi_1 \wedge \cdots \wedge \psi_l)$$

where ψ_i are atomic, i.e. of the form $R(x_{i_1}, \ldots, x_{i_k})$ for $R \in \tau$.

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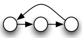
Example:



not homomorphic to $(\mathbb{Q}; <)$.

$\exists x_1, x_2, x_3 \, (x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1)$ is false in $(\mathbb{Q}; <)$.

More Examples of CSPs

Betweenness:

Input: A finite set V, and a subset S of V^3.

Question: Is there a linear order $<$ on V such that for every $(u, v, w) \in S$ we have $u < v < w$ or $w < v < u$?

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Complexity: in P

CSPs of Open Complexity

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Strongest evidence comes from the so-called universal algebraic approach.

Primitive Positive Definability

Lemma (Jeavons et al'97).

Let $\Gamma = (D; R_1, \ldots, R_k)$ be a relational structure, and
let R be a relation that has a primitive positive definition in Γ.
Then $\mathrm{CSP}(\Gamma)$ and $\mathrm{CSP}(D; R, R_1, \ldots, R_k)$ are polynomial-time equivalent.

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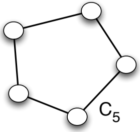

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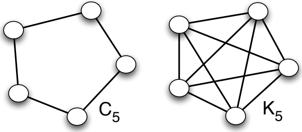

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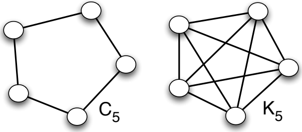

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$$E'(x,y) \equiv \exists p_1, p_2, p_3, q_1, q_2 \big(E(x,p_1) \wedge E(p_1,p_2) \wedge E(p_2,p_3) \wedge E(p_3,y) \wedge E(x,q_1) \wedge E(q_1,q_2) \wedge E(q_2,y)\big)$$

Polymorphisms

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A function $f\colon D^k \to D$ preserves $R \subseteq D^m$ if $\big(f(a_1^1,\dots,a_1^k),\dots,f(a_m^1,\dots,a_m^k)\big) \in R$ whenever $(a_1^i,\dots,a_m^i) \in R$ for all $i \leq k$.

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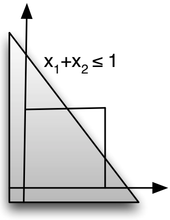
Example: $(x, y) \mapsto max(x, y)$ preserves a linear half-space given by $a_1 x_1 + \cdots + a_n x_n \leq a_0$ iff at most one of $a_1, \dots, a_n$ is positive.



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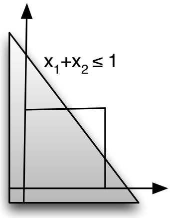

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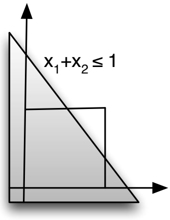

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Example: Every structure Γ has the projections as polymorphisms.

Polymorphisms and Primitive Positive Definability

Equivalent definition:
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Theorem (Geiger'68, Bodnarcuk et al'69).

Let Γ be finite. Then a relation R has a primitive positive definition in Γ if and only if R is preserved by all finitary polymorphisms of Γ.

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Polymorphisms $\leftrightarrow$ Algorithms

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Assume that Γ has a finite domain D.

Theorem (Bulatov+Jeavons+Krokhin'05,Maroti+McKenzie'08).

Let Γ be a finite structure. Then Γ has a weak near unanimity polymorphism of arity $n \geq 2$, this is, a polymorphism f such that for all elements x, y of Γ

$$f(y,x,\dots,x) = f(x,y,\dots,x) = \cdots = f(x,\dots,x,y) \; ,$$

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or $\mathrm{CSP}(\Gamma)$ is NP-hard.

Example:

$$(x, y) \mapsto max(x, y)$$

is a weak near unanimity polymorphism of $(\mathbb{Q}; <)$.

The Finite-Domain Tractability Conjecture

Bulatov+Jeavons+Krokhin'04 (in different, but equivalent form):

Conjecture 2.

If Γ has a weak near unanimity polymorphism, then $\mathrm{CSP}(\Gamma)$ is in P.

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There are two algorithmic techniques to obtain those results:

- Generalizations of Gaussian elimination
  (works for example when Γ has Maltsev polymorphism)
- 'Constraint Propagation' / Datalog

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   E.g. systematically study CSPs over the integers, rationals, and reals.

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General Goal:

for interesting base structures Δ, classify $\text{CSP}(\Gamma)$ for all reducts Γ of Δ.

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Theorem (B+Nešetřil'03).

Let Γ be ω-categorical. Then a relation R has a primitive positive definition in Γ if and only if R is preserved by all polymorphisms of Γ.

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- STACS Proceedings: tractability conjecture for a large class of ω-categorical structures Γ.

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- CSP$(\Delta, \{(u,v,x,y) \mid u = v \Rightarrow x = y\})$ is in P (Bäckström,Jonsson'98).

Tractable Expansions of Linear Programming

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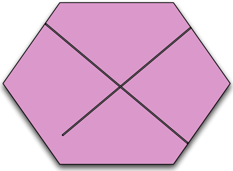
if for all $a, b \in R$ there are only finitely many points on the line segment between a and b that are not in R.



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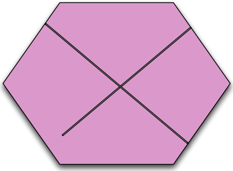

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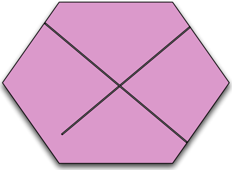

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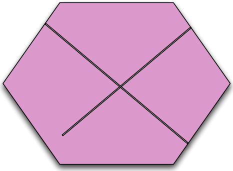

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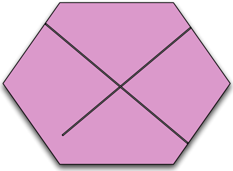

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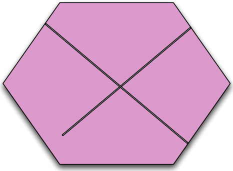

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Remarks.

- Essential convexity is not a polymorphism condition.
- But: essential convexity is a polymorphism condition in a saturated elementary extension of Γ (B.,Mamino'14).

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Revisit

$$\mathrm{CSP}(\mathbb{Q}; S_{+1}, S_{\cdot 2}, S_{max})$$

where

- $S_{+1} := \{(x, y) \mid y = x + 1\}$,
- $S_{\cdot 2} := \{(x, y) \mid y = 2x\}$, and
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Theorem (Möhring, Skutella, Stork'04).

Mean payoff games are polynomial-time equivalent to deciding satisfiability of constraints of the form $x \leq max(y, z) + c$ where $c \in \mathbb{Z}$ is represented in binary.

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Theorem (B,Martin,Mottet'15).

Let Γ be a reduct of $(\mathbb{Z}; \text{succ})$. Then $\text{CSP}(\Gamma)$ is in P, or NP-complete, or equals a finite-domain CSP.

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- Need to modify Γ and use more coding tricks so that CSP(Γ) is polynomial-time equivalent to "$n \in U$?" ...

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  These problems are all in NP.