Hoare Calculation and its Application

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A first example

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    result(x-1,y+1);
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As easy as in the first example?

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`result_2(x+1,y)`

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<u>Try to prove the assertion by induction on x:</u>

$x = 0 \Rightarrow$ `result_2(0,y)` $= y$ and $y = y + x$ ✓

Let the assumption be proved for some $x \in \mathbb{N}_0$ and all $y \in \mathbb{N}_0$.
`result_2(x+1,y)` $= \ldots$ It doesn't work!

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- Number of `while`-loop-iterations depends on x.
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$\Rightarrow$ Mathematical methods of proof won't last!

$\Rightarrow$ We need new tools!

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In the following algorithms the termination is assumed.

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- Both tasks are as hard as the Halting Problem.
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In the following algorithms the termination is assumed.
⇒ We just meet challenge 2 using Hoare Calculation . . .

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> *"I conclude that there are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies and the other way is to make it so complicated that there are no obvious deficiencies."*

Hoare-Triple

$$\{\mathtt{P}\}\ \mathbf{S}\ \{\mathtt{Q}\}$$

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$\{\texttt{P}\}\ \mathbf{S}\ \{\texttt{Q}\} = \texttt{true} :\Leftrightarrow$
If the predicate $\{\texttt{P}\}$ is true immediately before execution of **S**, then immediately **S** has terminated, the predicate $\{\texttt{Q}\}$ is true.

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Notation: $\frac{X}{Y} :\Leftrightarrow X \Rightarrow Y$

Hoare Rule 1: Skip-Axiom

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skip means the program with no commands.

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$\texttt{A}_{\beta/x}$ is predicate $\texttt{A}$, but x instead of β.

Hoare Rule 3: Rule of Composition

$$\frac{\{\mathtt{A}\}\ \mathbf{S1}\ \{\mathtt{B}\} \wedge \{\mathtt{B}\}\ \mathbf{S2}\ \{\mathtt{C}\}}{\{\mathtt{A}\}\ \mathbf{S1},\mathbf{S2}\ \{\mathtt{C}\}}$$

Hoare Rule 4: Rule of Conditional Branching

$$\frac{\{\mathtt{A} \wedge \mathtt{B}\}\ \mathbf{S1}\ \{\mathtt{Q}\} \wedge \{\mathtt{A} \wedge \neg\mathtt{B}\}\ \mathbf{S2}\ \{\mathtt{Q}\}}{\{\mathtt{A}\}\ \mathbf{if}\ \mathtt{B}\ \mathbf{then}\ \mathbf{S1}\ \mathbf{else}\ \mathbf{S2}\ \mathbf{end\ if}\ \{\mathtt{Q}\}}$$

Hoare Rule 5: Rule of Iteration

$$\frac{\{\mathtt{I} \wedge \mathtt{B}\}\ \textbf{S}\ \{\mathtt{I}\}}{\{\mathtt{I}\}\ \textbf{while}\ \mathtt{B}\ \textbf{loop}\ \textbf{S}\ \textbf{end loop}\ \{\mathtt{I} \wedge \neg\mathtt{B}\}}$$

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Such an `I` is called loop-invariant.

Hoare Rule 6: Rule of Consequence

$$\frac{\mathtt{A} \Rightarrow \mathtt{A'} \quad \wedge \{\mathtt{A'}\}\ \mathtt{S}\ \{\mathtt{B'}\} \wedge \quad \mathtt{B'} \Rightarrow \mathtt{B}}{\{\mathtt{A}\}\ \mathbf{S}\ \{\mathtt{B}\}}$$

Proof of `result_2(x,y)` using Hoare

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    {I $\wedge \neg$ B}  (*Rule of Iteration*)
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- B: $x > 0$ (condition in `while`-loop)

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- loop-invariant:

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- B: $x > 0$ (condition in `while`-loop) $\Rightarrow \neg$ B: $\neg(x > 0)$
- loop-invariant: I: $r = x + y$

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$\{\mathrm{I}\colon r = x + y \wedge \mathrm{B}\colon x > 0\}$

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while x > 0
   {I: $r = x + y \wedge$ B: $x > 0$}
   {Item 1}
   x = x-1;
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- Rule of Assign.:

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  x = x-1;
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  {I: $r = x + y$, $x \geq 0$}
```

- Rule of Assign.: Item 2: $\{x + (y + 1),\ x \geq 0\}$

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- Rule of Assign.: Item 2: $\{x + (y + 1),\ x \geq 0\}$
- Assign.:

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- Rule of Assign.: Item 2: $\{x + (y + 1), x \geq 0\}$
- Assign.: Item 1: $\{(x - 1) + (y + 1), x - 1 \geq 0\}$

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- Rule of Assign.: Item 2: $\{x + (y + 1),\ x \geq 0\}$
- Assign.: Item 1: $\{(x - 1) + (y + 1),\ x - 1 \geq 0\}$
- In fact:

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- Assign.: Item 1: $\{(x - 1) + (y + 1),\ x - 1 \geq 0\}$
- In fact: $\{(x - 1) + (y + 1),\ x - 1 \geq 0\}$

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- Rule of Assign.: Item 2: $\{x + (y + 1),\ x \geq 0\}$
- Assign.: Item 1: $\{(x - 1) + (y + 1),\ x - 1 \geq 0\}$
- In fact: $\{(x - 1) + (y + 1),\ x - 1 \geq 0\} \triangleq \{$I: $x + y \wedge$ B: $x > 0\}$

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```

- Rule of Assign.: Item 2: $\{x + (y + 1),\ x \geq 0\}$
- Assign.: Item 1: $\{(x - 1) + (y + 1),\ x - 1 \geq 0\}$
- In fact: $\{(x - 1) + (y + 1),\ x - 1 \geq 0\} \triangleq \{$I: $x + y \wedge$ B: $x > 0\}$ because $x - 1 \geq 0 \Leftrightarrow x > 0$ for integer x.

```
function result_2(x,y)
    {P: $x \geq 0 \wedge y \geq 0$, $r := x + y$}
    {I: $r = x + y$}
    while x > 0
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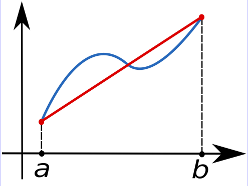

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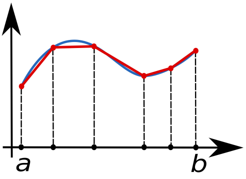

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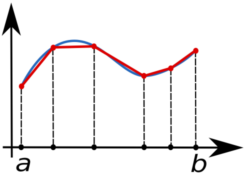

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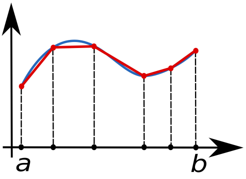

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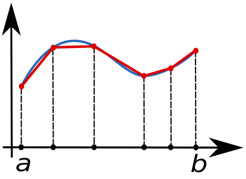

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$$\Delta F \leq \frac{(b-a)^3}{12 \cdot n^2} \cdot ||f''||_\infty$$

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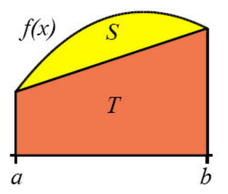

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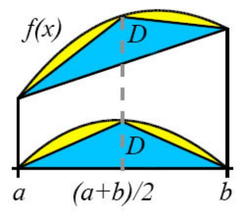

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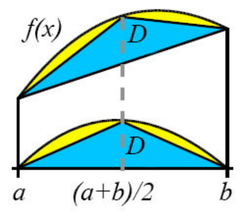

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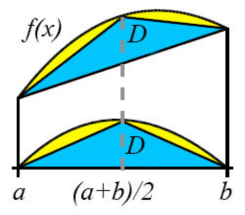

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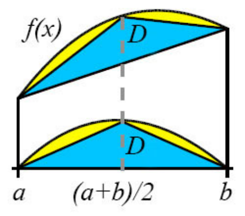
$$D(f, a, b) = \frac{b-a}{2} \cdot h$$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D
D
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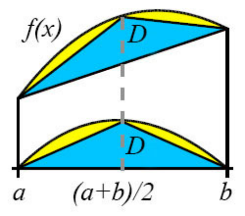
(a+b)/2
b



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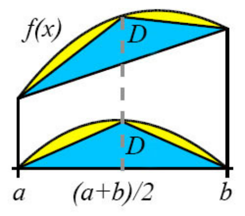

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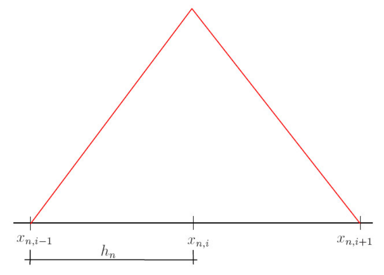

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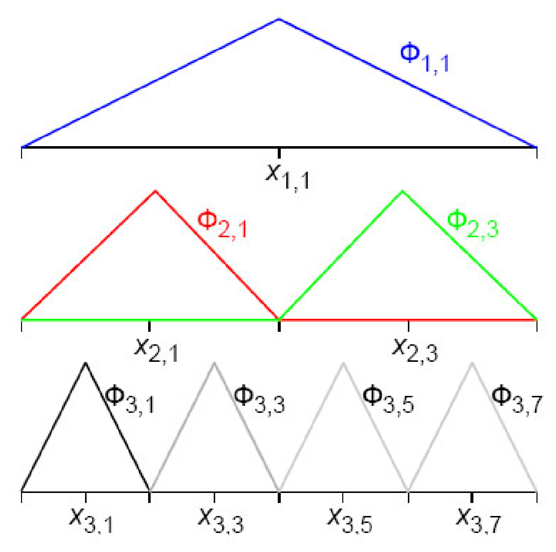
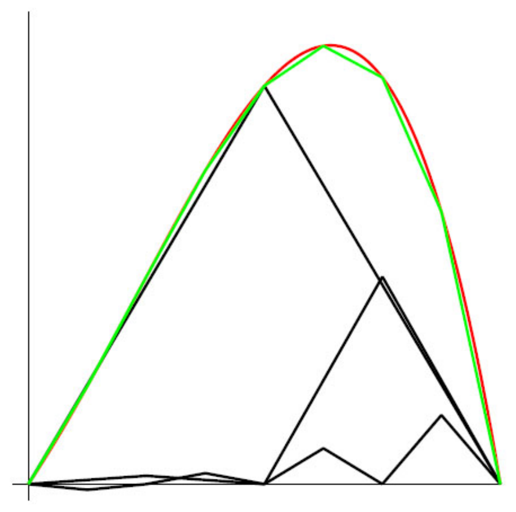
The hierachical basis for W_1, W_2 and W_3



Approximation



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The program `HierachicalBasis(N)` should convert $v(x)$ into the hierachical basis: $(N > 1)$

Representation in hierachical basis

Let $v \in V_N$ be a vector:

$$v(x) = \sum_{n=1}^{N} \sum_{i=1}^{2^n-1} \alpha_{n,i} \Phi_{n,i}(x)$$

The program `HierachicalBasis(N)` should convert $v(x)$ into the hierachical basis: $(N > 1)$

$$v(x) = \sum_{n=1}^{N} \sum_{i=1}^{2^n-1} \alpha'_{n,i} \Phi_{n,i}(x)$$

with $\alpha'_{n,i} = 0$ for all even i.

Program `HierachicalBasis(N)`

```
function HierachicalBasis(N)
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    for i = 1,...,$2^n - 1$ :
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Program HierachicalBasis(N)

```
function HierachicalBasis(N)
  for n = N-1,...,1 :
    for i = 1,...,$2^n - 1$ :
      $a_{n+1,2i-1}$ -= $a_{n+1,2i}/2$
      $a_{n+1,2i+1}$ -= $a_{n+1,2i}/2$
```

Program `HierachicalBasis(N)`

```
function HierachicalBasis(N)
  for n = N-1,...,1 :
    for i = $1,\ldots,2^n-1$ :
      $a_{n+1,2i-1}$ - = $a_{n+1,2i}/2$
      $a_{n+1,2i+1}$ - = $a_{n+1,2i}/2$
      $a_{n,i}$ + = $a_{n+1,2i}$
```

Program HierachicalBasis(N)

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function HierachicalBasis(N)
  for n = N-1,...,1 :
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      $a_{n+1,2i-1}$ -= $a_{n+1,2i}/2$
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      $a_{n,i}$ += $a_{n+1,2i}$
      $a_{n+1,2i}$ = $0$
```

Program `HierachicalBasis(N)`

```
function HierachicalBasis(N)
  for n = N-1,...,1 :
    for i = 1,...,$2^n - 1$ :
      $a_{n+1,2i-1}$ -= $a_{n+1,2i}/2$
      $a_{n+1,2i+1}$ -= $a_{n+1,2i}/2$
      $a_{n,i}$ += $a_{n+1,2i}$
      $a_{n+1,2i}$ = 0
```

To prove the correctness of `HierachicalBasis(N)`, the programm must be written in a form Hoare Calculation can handle with:

```
function HierachicalBasis_Hoare(N)
```

```
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  n = N-1
  while n $\neq$ 0
```

```
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  n = N-1
  while n $\neq$ $0$
    i = 1
    while i $\neq$ $2^n$
```

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  n = N-1
  while n $\neq 0$
    i = 1
    while i $\neq 2^n$
      $a_{n+1,2i-1} = a_{n+1,2i-1} - a_{n+1,2i}/2$
      $a_{n+1,2i+1} = a_{n+1,2i+1} - a_{n+1,2i}/2$
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      $a_{n+1,2i} = 0$
```

```
function HierachicalBasis_Hoare(N)
  n = N-1
  while n $\neq$ 0
    i = 1
    while i $\neq$ $2^n$
      $a_{n+1,2i-1} = a_{n+1,2i-1} - a_{n+1,2i}/2$
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      $a_{n,i} = a_{n,i} + a_{n+1,2i}$
      $a_{n+1,2i} = 0$
      i = i+1
    n = n-1
```

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$\Rightarrow$ Hoare Calculation!

References

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End of presentation

Thank you for your attention!