The Turbo Principle in Communication Systems

Introduction

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Outline Basic Turbo Techniques

Introduction to Serial Decoding

Introduction to Parallel Decoding (Turbo Codes)

The BCJR Algorithm

Log-Likelihood Ratios (LLR) and the APP Decoders

The Turbo Principle and its Applications
Outline Applications

Conclusions

- Turbo Applications: Turbo Source Compression
- Turbo Applications: Analog Turbo Decoders
- Turbo Applications: Pre-coded QAM with Irregular Channel Codes
- Turbo Applications: Source Channel Coding for Continuous Sources
- Turbo Applications: Source Channel Coding with Variable Length Codes (VLC)
- Turbo Applications: Coded MIMO Systems
- Turbo Applications: Coded Equalization of Multipath Channels
Introduction

History:

1948: Shannon's absolute limits in communications, e.g. $0.2 \text{dB}$ in $E_b/N_0$ for binary codes with rate $1/2$ on AWGN channel.

1962: Gallager's low density parity check codes with iterative decoding.

1966: Forney: Concatenated codes (Viterbi and RS codes) approach Shannon's limit by $2.5 \text{dB}$.

1977: Turbo principle recognized as general method in communications systems.


2001: Chung, Forney, Richardson, Urbanke: Iterative decoding of LDPC codes within $0.0045 \text{dB}$ of Shannon limit.
The Turbo Principle comprises:

- a communication system with serial and/or parallel concatenations of components
- a posteriori probability (APP) symbol-by-symbol decoders/detectors
- soft-in/soft-out decoders/detectors
- interleavers between the components
- exchange of extrinsic information between components in the form of probabilities or log-likelihood ratios
The Turbo Principle...
The Turbo Principle...
Examples for serial concatenation in communications systems.
Examples for standardized applications for the turbo principle systems:

- LDPC for ESA Standard Digital Video Broadcasting (DVB)
- NASA Deep Space standard
- IEEE 802 Wireless LAN
- Rate 1/3 PCC Code in UMTS
The Turbo Principle in Communication Systems

Log-Likelihood Values and APP Decoders

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(Short Course 2004)
Let $u$ be in $\mathbb{F}_2$. The log-likelihood ratio (LLR) or L-value of the binary variable is

$$L(u) = \ln \frac{P(u = 1)}{P(u = 0)}$$

with the inverse

$$\frac{1 - n}{1 + n} u | = (n)^T$$

The reliability of this decision is $|(n)^T|$. The sign of $(n)^T$ is the hard decision and the magnitude is $\|L(u)\|$. Note: The sign of $(n)^T$ is the hard decision and the magnitude is $\|L(u)\|$. The elements under the addition $\oplus$ are in $\mathbb{F}_2$. Let $n$ be in $\mathbb{F}_2$. The elements of the binary variable is

$$\mathbb{F}_2 = \{0, 1\}$$

where $+1$ is the null element.
The soft bit and the binary sum

The soft bit ($u$) is

\[
E[u] = \tanh(L(u)) = \tanh(2) \cdot (u_1 + 2) \cdot (u_2 + 2) = \{n\} \cdot \{1\} = \{n\} \\
\]

with the boxplus abbreviation.

\[
(u) \boxplus (1) = ((2/(u)) \cdot \tanh \cdot (2/(1)) \cdot \tanh) \cdot (2) = (2n \oplus 1n) \\
\]

The soft bit is $\{n\} \cdot \{1\}$. 

GF(2) addition of two independent binary random variables:

\[
E(u_1 \cdot u_2) = \{n\} \cdot \{1\} = \{n\} \\
\]

The soft bit and the binary sum
The boxplus element and its approximation (1)

The boxplus element

\[ L(u_1 \oplus u_2) = 2 \tanh^{-1} \left( \tanh(L(u_1)/2) \cdot \tanh(L(u_2)/2) \right) \]

can be approximated by

\[ L(u_1) \boxplus L(u_2) \approx \frac{\text{sign}(L(u_1)) \cdot \text{sign}(L(u_2)) \cdot \min\{|L(u_1)|, |L(u_2)|\}}{L(u)} \]

Examples:

- \(-3.0 \oplus +0.5 \approx -0.5\)
- \(-4.0 \oplus -1.2 \approx +1.2\)
- \(L(u) \oplus +\infty = L(u)\)
The boxplus element and its approximation (2)

\[ (|\mathfrak{t}_{-1}\mathfrak{t}| - \mathfrak{t} + 1)\mathfrak{t} + \{\mathfrak{t}_{-1}\mathfrak{t} \} \max = (\mathfrak{t}_{-1}\mathfrak{t} + \mathfrak{t}_{+1}\mathfrak{t})\mathfrak{t} \]

A similar approximation is known as the max operation (Jacobian Loga-

maximum value of \[ \mathfrak{t} \].

appears. It is necessary when both magnitudes are the same and has a

If one of the two magnitudes is dominant the correction term dis-

\[ \frac{| |(\mathfrak{t}_{-1}\mathfrak{t})| + |(\mathfrak{t}_{+1}\mathfrak{t})| - \mathfrak{t} + 1 |}{| |(\mathfrak{t}_{-1}\mathfrak{t})| - |(\mathfrak{t}_{+1}\mathfrak{t})| - \mathfrak{t} + 1 |} \mathfrak{t} - \]

\[ \{ |(\mathfrak{t}_{-1}\mathfrak{t})| , |(\mathfrak{t}_{+1}\mathfrak{t})| \} \max \cdot \{(\mathfrak{t}_{-1}\mathfrak{t})\mathfrak{t} \max \cdot \{(\mathfrak{t}_{+1}\mathfrak{t})\mathfrak{t} \max = (\mathfrak{t}_{-1}\mathfrak{t} \oplus (\mathfrak{t}_{+1}\mathfrak{t})\mathfrak{t} \]

\[ = \{(\mathfrak{t}/(\mathfrak{t}_{-1}\mathfrak{t})\tan \mathfrak{t}) \tan \mathfrak{t}, (\mathfrak{t}/(\mathfrak{t}_{+1}\mathfrak{t})\tan \mathfrak{t}) \tan \mathfrak{t} \} \max = (\mathfrak{t}_{-1}\mathfrak{t} \oplus (\mathfrak{t}_{+1}\mathfrak{t})\mathfrak{t} \]

The boxplus element and its approximation (2)
The binary XOR, the boxplus and the softbit operation:

\[(z \cdot 1)_T = (z/\tanh^{-1}(z/1) + 1)_T = (z \oplus 1)_T\]

The boxplus element.
Transmission and combining after fading/AWGN channels

The a posteriori probability (APP) in $y = ax + n$ is

$$\text{P} (x | y) = \frac{\text{p}(y | x) \text{P}(x)}{p(y)} \quad (3)$$

with the pdf $p(y | x) = \frac{\gamma^2}{2} \exp\left(-\frac{(y - ax)^2}{2\gamma^2}\right) \quad (4)$

The complementary APP $\text{LLR}$ equals

$$\text{L}c = 2 \frac{\sigma^2}{\gamma^2} = 4 \frac{\sigma^2}{\gamma^2} = 4 \frac{\sigma^2}{\gamma^2} \quad (5)$$

For statistically independent transmission

$$\text{L}(x) = \text{L}(x | y_1; y_2) = \text{L}(x | y_1) + \text{L}(x | y_2) + \text{L}(x) \quad (7)$$

For statistically independent transmission

$$\text{N} / s \mathcal{P} \mathcal{V} = \frac{\sigma^2}{\nu \gamma} = \gamma \text{T} \quad (6)$$

The complementary APP $\text{LLR}$ equals

$$\frac{\gamma^2}{\varphi\gamma^2} = \text{L}(x | y) \quad (8)$$

The complementary APP $\text{LLR}$ equals

$$\frac{(f_i | y)}{(x) | y} = (f_i | x) \quad (9)$$

With the pdf

$$\frac{(f_i | y)}{(x) | y} = (f_i | x) \quad (10)$$

For statistically independent transmission

$$\text{L}(x) = \text{L}(x | y_1; y_2) = \text{L}(x | y_1) + \text{L}(x | y_2) + \text{L}(x) \quad (7)$$

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With the pdf

$$\frac{(f_i | y)}{(x) | y} = (f_i | x) \quad (10)$$
Practical Usefulness of Log-Likelihood Calculation

Did it rain in NY at 1:00 pm today?

A Yes (rain!) is binary coded as +1, transmitted over an unreliable link. Additional a priori value is available: From Farmer’s Almanac:

\[
\begin{array}{cccc}
+1 & 0.9 & 0.25 & +1.1 \\
+2.7 & 3.0 & 0 & 1.0 \\
-3.0 & 2.0 & -1.5 & 0.9 \\
\end{array}
\]

Probability of rain in NY today is 75% with 31% error: rain in NY!!

For statistically independent information:

\[
L(x) = \ln \left( \frac{0.75}{0.25} \right) = (x)^T + \delta
\]

\[
L_c = \begin{cases}
0 & \text{if } x = +1 \\
1 & \text{if } x = -1 \\
2 & \text{if } x = \text{channel state}
\end{cases}
\]

Transmitted value \( x \) received value \( y \)

\[
L_c = \begin{cases}
0 & \text{if } y = 0 \\
1 & \text{if } y = 1 \\
2 & \text{if } y = \text{link 1 or link 2}
\end{cases}
\]

For statistically independent information:

\[
L(x) = \ln \left( \frac{0.75}{0.25} \right) = (x)^T + \delta = \ln(1.25) + 1 = 1.1
\]

Two rain detection devices measured:

\[
\begin{array}{cccc}
+1 & 0.9 & 0.25 & +1.1 \\
+2.7 & 3.0 & 0 & 1.0 \\
-3.0 & 2.0 & -1.5 & 0.9 \\
\end{array}
\]

\[
L_c = \begin{cases}
0 & \text{if } x = +1 \\
1 & \text{if } x = -1 \\
2 & \text{if } x = \text{channel state}
\end{cases}
\]

Transmitted value \( x \) received value \( y \) channel state \( \delta \)
The extrinsic information as a LLR

Assume a parity check equation of statistically independently transmitted bits

Then the extrinsic LLR for the first bit is

\[ L_e(x_1) = (\frac{\partial f}{\partial x} x_1)^T \]

With the APF LLR's

\[ (\frac{\partial f}{\partial x} x_1 y_1) \]

Example:

SPC code, \( N = 3 \), with \( L_e(x_1) = 0 \):

\[ (\frac{\partial f}{\partial x} x_1)^T = (x)^T \]

Then the extrinsic bit \( x \) equals

\[ 0 = (\frac{\partial f}{\partial x} x) \oplus \frac{1}{N} \]

The extrinsic information is statistically independently transmitted
The general formula for a soft output as a LLR

Assume a transmission of a vector \( x \) over a channel and received as a vector \( y \).

We are interested in the LLR of the \( n \)-th bit of \( x \).

The metric can be expanded in channel and a priori parts

\[
(6) \quad (\Lambda|d)_{\overline{1}} - (x|d)_{\overline{1}} + (x|\Lambda)_{\overline{1}} = (\Lambda|x)_{\overline{1}}
\]

\[\text{The metric can be expanded in channel and a priori parts}\]

\[
(8) \quad \frac{(\Lambda|x)_{\overline{1}}}{(\Lambda|x)_{\overline{1}}} \cdot \frac{1-\epsilon^{u x \leq}}{1+\epsilon^{u x \leq}} = \frac{(\Lambda|x)_{\overline{1}}}{(\Lambda|x)_{\overline{1}}} \cdot \frac{1-\epsilon^{u x \leq}}{1+\epsilon^{u x \leq}} = (\Lambda|u x)_{\overline{1}} = (u x)_{\overline{1}}
\]

Note: If all paths have the same length, we can ignore the last term.

However, if we search paths in a tree with different length of the paths, we cannot ignore the term during the search.
The channel part of the metric

Transmission over a real AWGN or multiplicative fading channel with

\[ u_n x \cdot u_\phi \cdot u_D \cdot \sum_{i=0}^{N} \frac{1}{H} \frac{1}{Y} = (u_n x | u_\phi) \frac{1}{X} \frac{1}{Y} \]

is the correlation metric

\[ (x | X) \frac{1}{Y} \]

The receiving process is corrupted by real valued AWGN with i.i.d. noise

\[ u_n m + u_n x \cdot u_D = u_\phi \]

The channel part of the metric

After cancellation of all parts common in the denominator and denominator

\[ \frac{1}{H} \frac{1}{Y} = \frac{m_0}{2} \]

with

\[ \frac{m_0}{2} \frac{1}{Y} = (m) d \]

samples and pdf
The Channel Part of the Metric

Transmission over an intersymbol interference (ISI, multipath) channel with L-taps

\[ y[n] = \sum_{i=0}^{L-1} h[i] \cdot x[n-i] + w[n] \]

This leads to the channel part of the metric

\[ p(y|x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2} (y - \mu)^2} \]

The receiving process is corrupted by complex-valued AWGN with i.i.d. noise samples and pdf

The receiving process is corrupted by complex-valued AWGN with i.i.d.

\[ (10) \]

\[ u[n] + \sum_{i=1}^{L} y[n-i] = u[n] \]

L-taps
The A Priori Part of the Metric

With the statistical independence from the interleaver we have

\[ \ln P(x) = \sum_{n=1}^{N} \ln P(x_n) \]

With the L-values we obtain

\[ \ln P(x_n) = \ln P(x_n) + L(x_n) \]

This leads to the soft output

\[ L(\hat{x}_n) = L(x_n) + \ln P(x_n) \]

where the last two terms can be deleted if all paths have equal length.

With the L-values we obtain

\[ L(x) \]

With the statistical independence from the interleaver we have

\[ L(x) = \ln P(x) \]
The APriori Part, the Channel Part and the Extrinsic Part

With the fading or AWGN channel we obtain finally from

\[
\frac{z/(f x) T f x + (f x|fh)}{\sum_{\mathcal{N}}^{\text{max}} -z/(f x) T f x + (f x|fh)} \begin{cases} 
1 & \text{if } u \neq f \text{ and } \mathcal{I} = f, I = -u x, \\
\text{max} & \text{if } u \neq f \text{ and } \mathcal{I} = f, I = +u x \\
0 & \text{otherwise}
\end{cases}
\]

the so-called max-log approximation for the extrinsic part.

With the fairly tight approximation \( \text{max} = \mathcal{I} \wedge \mathcal{I} \), we obtain the

The last part is called the extrinsic part of the soft-output. It represents the

influence of all the other bits on the current bit with index \( u \).

With the fading or AWGN channel we obtain finally from

The APriori Part, the Channel Part and the Extrinsic Part
The Turbo Principle in Communication Systems

The APP Decoder on a Trellis with the Bahl-Cocke-Jelinek-Raviv (BCJR) Algorithm

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(Short Course 2004)
The APP Decoder on a Trellis: The BCJR Algorithm

The algorithm is due to Bahl-Cocke-Jelinek-Raviv based on earlier work by Welsh and Baum.

For a binary trellis let $S_i$ be the encoder state at time $k$. The bit is $y_n$.

The algorithm is to provide us with

The index pair $s$ and $s'$ determine the information bit and the coded bits.

The sum of the joint probabilities in (11) is taken over all existing transitions from state $s$ to state $s'$ labeled with the information bits. The goal of the MAP algorithm is to provide us with $L_n(u_k)$.

For a binary trellis, let $S_i$ be the encoder state at time $k$. The bit is $y_n$.

The algorithm is due to Bahl-Cocke-Jelinek-Raviv based on earlier work by Welsh and Baum.

The algorithm is to provide us with
Assuming a memoryless transmission channel, the joint probability

\[
\Pr(s_0 \mid y_{\leq k} \cup y\mid s_0) \Pr(y_k \mid s_0, s) \Pr(y_{k+1} \mid s) = \sum_{s} \Pr(s_0 \mid y_{\leq k} \cup y\mid s_0) \Pr(y_k \mid s_0, s) \Pr(y_{k+1} \mid s) 
\]

\[
= \sum_{s} \Pr(s_0 \mid y_{\leq k} \cup y\mid s_0) \Pr(y_k \mid s_0, s) \Pr(y_{k+1} \mid s)
\]

 intimates following BCJR 1974.'
The forward recursion of the MAP algorithm yields

\[ k(s) = \sum_{s_0} k(s_0; s) \]

(12)

The backward recursion yields

\[ k(s_0) = \sum_{s} k(s_0; s) \frac{P(s)}{P(s_0)} \]

(13)

The branch transition probabilities are given by

\[ k(s_0; s) = p(y_k | u_k) \cdot P(u_k) \]

(14)
The Transition Metrics as LLR

Using the log-likelihoods the a priori probability can be expressed as $\Pr(\gamma n|\gamma K)$.

The Transition Metrics as LLR

and hence will cancel out in the ratio of (11).

for a binary input multipath channel with $L + T$ taps. The terms $A^k$ and $B^k$ in (15) and (16) are equal for all transitions from level $k - 1$ to level $k$ and

\[
\Pr(\gamma n|\gamma K) = \frac{\sum_{k} A^k B^k}{\sum_{k} A^k}
\]

and in a similar way, the conditioned probability

\[
(\frac{\gamma n}{\gamma n})^T \gamma n^\Theta \cdot \gamma A = \frac{\gamma n}{\gamma n}^T \gamma n^\Theta \cdot \left(\frac{\gamma n}{\gamma n}^T - \Theta + \frac{\gamma n}{\gamma n}^T + \Theta\right) = (\gamma n)^P
\]
A Simplification of the BCJR-Algorithm

Approximation of the BCJR algorithm is given by using the approximation

\[ \log X_i e^{L_i \max} = (s)^{1 - \gamma_0} W \]

and for the backward algorithm

\[ \{(s', s) e^{L_0} \log \} \max = (s)^{\gamma_0} W \]

For the forward algorithm we get

\[ (s) e^{L_0} \log = (s)^{\gamma_0} W \]

for the backward algorithm

\[ (s)^{1 - \gamma_0} \log = (s)^{1 - \gamma_0} W \]

They produce the state metrics for the forward algorithm terminate trellis. They produce the state metrics for the forward algorithm mutate into two Viterbi algorithms running forth and back the algorithm and for the backward algorithm

\[ \log \approx \sum e^{L_0} \log \]

An approximation of the BCJR algorithm is given by using the approximations of the BCJR algorithm.
Using again the approximation (18) the soft-output results in

\begin{align*}
\log p(y_k) &= \max \left( s_0, s \right) - \left( 1 - |\gamma K| \right) d \delta_0 \\
&= \max \left( s_0, s \right) + (s)_{1-\gamma K}^I - \left( s_0, s \right)_{1-\gamma K}^I \\
&= (s_0, s)_{1-\gamma K}^I \max\left( s_0, s \right) = (\gamma \eta) T \quad \text{for } s_0 = 1
\end{align*}
The soft-output of the simplification with the feedforward trellis

For a binary trellis three different butterfly structures exist.

For the structure where the two paths with same $\gamma n$ merge in one state $s$ –

this is the case for feedforward convolutional codes and tapped delay line channels –

the first three terms in (24) form (22) and the maximization is only over the states $s$:

\[
\begin{align*}
\forall n: \\
(s)^{\gamma n} I_W + (s)^{\gamma n} I_W \quad \text{max}_{s}^{1-\gamma n} \quad & - \\
(s)^{\gamma n} I_W + (s)^{\gamma n} I_W \quad \text{max}_{s}^{1+\gamma n} \quad &= (\gamma n) T
\end{align*}
\]
The simplified BCJR Algorithm

The BCJR algorithm for mostly used binary terminated trellises can be closely approximated by two VA algorithms running backwards and forwards using the update metric 

\[ \frac{1}{2} \left( \frac{3}{2} n \right) T \log \left( \frac{1}{2} \right) \]

where \( c \) is a suitable simplifying normalization constant independent of \( n \). Therefore, if the SNR is very small the soft-output equals only the a priori value as it should be. Note, that the channel part of the update metric has the SNR as a factor.

Obtain the soft output.

Find the maxima over the plus and minus states and subtract them to obtain the soft output.

\[ \frac{c}{n} \left( \frac{3}{2} n \right) T \log \left( \frac{1}{2} \right) \]

- Find the current bit.
- Add the forward- \((\mathcal{F})\)- to the backward- \((\mathcal{B})\)- metrics to the right \((k)\) of \( u_k \).

A memory storing the metrics.

Using the update metric \( \log \left( \frac{1}{2} \right) \) of two VA algorithms running backwards and forwards.

The simplified BCJR Algorithm

Two VA algorithms for the mostly used binary terminated trellises can be closely approximated by...
Several variations of the SOVA exist:

- It delivers too optimistic (too large) L-values.
- It has the lowest complexity of all soft-in/soft-out algorithms.
- It is an add-on feature to the VA and can be turned on and off for individual bits.

Battail algorithm.

Further suboptimal and simplified soft-in/soft-out algorithms...
Parallel Concatenation

The Turbo Principle in Communication Systems

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Principle of Turbo Decoding for a parallel concatenated scheme.

Exchange of extrinsic information between horizontal and vertical decoding
A First Example of PCC Turbo Decoding
The Performance of the First Example of PCC Turbo Decoding

The graph shows the BER (Bit Error Rate) performance of a parallel concatenated SPC Code with a rate of 1/2. The Shannon limit is also indicated on the graph. The graph includes curves for different iterations (it. 0, it. 1, it. 1−4) and the uncoded performance. The horizontal axis represents BER, and the vertical axis represents $E_b/N_0$ in dB.
The general parallel concatenated code PCC Turbo Code
The principle of the Interleaver and Deinterleaver
The Decoder of the Parallel Concatenated Code PCC Turbo Code

Exchange of extrinsic information between horizontal (direct) and vertical (interleaved) decoding
Showing the chaotic behavior of Turbo decoding in a demonstration.

The turbo decoder decodes a parallel concatenated rate 1/2 code with memory 2, rate 2/3 convolutional code as constituent codes.

The display shows 20 x 20 interleaver matrix with half iteration. Shown are the soft-output L-values of the information bits after each decoder: SOVA algorithm with L-values.

Block interleaver: size 20 x 20 = 400 information bits, 800 transmitted.

Green circles for correct bits, Red Circles for wrong bits, Diameter of circles is the reliability (magnitude of L-values).

Goal of decoding: Big green circles!!!
Performance of Decoder of a PCC Turbo Code

Rate 1/2, constituent code: rate 2/3, memory 2, interleaver size 1024
Influence of the Interleaver Size for a PCC Turbo Code
The Turbo Principle in Communication Systems

Serial Concatenation

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Principle of Turbo Coding for a serially concatenated scheme.
The Decoder for a Serially Concatenated Scheme.
The Performance of a Serially Concatenated Scheme

Serial Concatenation of two small convolutional codes

$E_b/N_0$ in dB

BER

Shannon limit

it. 0

it. 1

it. 2

it. 3

it. 4

it. 4-9

uncoded $R=1/3$, $M=2$ recursive conv. code

$R=1/3$ serial concatenated code

unwanted
The Performance of a Serially Concatenated scheme

Serial Concatenation without iteration of an Outer Reed-Solomon Code (255,223) over GF($2^8$) and an Inner Convolutional Code, Rate 1/2 and Memory 6:

With iterations between the RS decoder and the convolutional decoder (state pinning of correctly decoded symbols) we achieved 1 dB further gain!!!

Oer, Hagenauer ICC 1993.
<table>
<thead>
<tr>
<th>Variable nodes</th>
<th>Check nodes</th>
<th>LDPC code/decoder</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEC en-/decoder</td>
<td>source encoder</td>
<td>turbo source-channel</td>
</tr>
<tr>
<td>DPSK 2 state trellis decoder</td>
<td>FEC en-/decoder</td>
<td>turbo DPSK</td>
</tr>
<tr>
<td>MIMO Mapper &amp; demapper</td>
<td>FEC en-/decoder</td>
<td>turbo MIMO</td>
</tr>
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<td>FEC en-/decoder</td>
<td>turbo BICM</td>
</tr>
<tr>
<td>MIMO multipath channel/detector</td>
<td>FEC en-/decoder</td>
<td>turbo equalization</td>
</tr>
<tr>
<td>FEC en-/decoder</td>
<td>FEC en-/decoder</td>
<td>serial code concat.</td>
</tr>
<tr>
<td>en-/decoder II (inner code)</td>
<td>en-/decoder I (outer code)</td>
<td>configuration</td>
</tr>
</tbody>
</table>

More Serially Concatenated Schemes
Serial concatenation of tailbiting convolutional codes and DPSK

- Serial concatenation of tailbiting convolutional codes and DPSK
- A block of information symbols is encoded without overhead by a tailbiting convolutional code (TBCC)
- Several codewords are then bit-wise interleaved
- DPSK Modulation is applied
- DPSK Modulation connected by an Interleaver ring
- View the system as a ring for the TBCC and another ring for the DPSK
- Demodulator and convolutional decoder operate in sequence
- When realized with digital processors

(see section Analog Decoding)
Serial concatenation of tailbiting convolutional codes and DPSK

\[
(1^{-1}x)T \oplus (x)T = (q)T
\]

Since \(x_i = b_i x_1\) and consequently at the receiver we can realize the DPSK decoder by

\[
L((q)x) = L((x)) + L((x)x_1)
\]

APP-Decoder for DPSK with forward and backward loop.
Serial concatenation of tailbiting convolutional codes and DPSK.

- TBCC decoder ring circuits.
- Which are connected via the interleaver ring to the
  APP-DPSK decoder segments $S^2$.
Serial concatenation of tailbiting convolutional codes and DPSK

Performance of the analog circuits: BER before decoder.

Time-continuous analog circuits better than 19 iterations with digital processors!!
A low density parity check (LDPC) code can be described as a serial concatenation of \( u \) variable nodes as inner repetition codes with \( n \) – \( k \) check nodes as outer single parity check nodes. Low Density Parity Check (LDPC) codes and their Turbo decoder.
The decoding result is the overall L value of the inner bits:

\[
L_{\text{(in)}}^{(i,j)} = \bigoplus_{p \neq \ell \neq 1} L_{\text{(c; out)}}^{(i,j)}
\]

where the \( \ell \)-th checks \( y \) are sent to the outer single parity check (SPC) per code bit \( x \).

More than one extrinsic message where the \( \ell \)-th checks \( y \) per variable node (\( \ell \)-th code bit) with connections. \( y - u \) - \( \ell \) is sent to the \( \ell \)-th decoder. The SPC decoders return for \( \ell \) = 1 \( \ldots, \ell \) is sent to the outer single parity check (SPC).

\[
L_{\text{(c; out)}}^{(i,j)} = \bigoplus_{p \neq \ell \neq 1} L_{\text{(c; in)}}^{(i,j)}
\]

\[
L_{\text{(out)}}^{(i,j)} = \bigoplus_{p \neq \ell \neq 1} L_{\text{(in)}}^{(i,j)} + \bigoplus_{p \neq \ell \neq 1} L_{\text{(c; out)}}^{(i,j)}
\]

Irregular LDPC codes and their Turbo decoder (cont.)
The Turbo Principle in Communication Systems

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Exercises

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(Short Course 2004)
Problem: Shannons Limit

Given the capacity formula:

$$C = \frac{1}{2} \log_2 (1 + 2^{E_s N_0})$$

Calculate the minimal $E_b = N_0$ in dB for rates $1/2$ and zero.

How do those limits change when you have a complex channel with in-phase and quadrature-phase?

\[ (\frac{0N}{s} + 1)^{\frac{3}{2}} \log_2 \frac{2}{1} = C \]
Problem: Boxplus Calculations

Calculate the following expressions using the box plus approximation:

1. $0.5 \oplus 1.5 = -1.5$
2. $0.8 \oplus 5.5 = 0.2$
3. $5.5 \oplus 0.2 = 5.5$

In which case is the approximation not good?
Problem: APP Decoding of a parity check code

A (3,4,2) parity check code is transmitted over an AWGN channel at an SNR of 3 dB. What is the probability the bit \( u_1 \) is in error?

Decoding.

Give the extrinsic and the APP LLR of the three information bits with APP decoding.

The received matched filter values \( y_i \) are +2.1, -1.3, -2.6, -4.5.

If it is known a priori that the first bit is with probability 0.75 a +1.

60
Problem: My first Turbo decoder

A parallel concatenated turbo code uses a total of 4 rate 2/3 SPC codes. A block interleaver is used. The following LLR values are received:

<table>
<thead>
<tr>
<th></th>
<th>+1.5</th>
<th>+0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>+5.0</td>
<td>-0.1</td>
</tr>
</tbody>
</table>

Perform one vertical and one horizontal iteration.

What is at this stage of iterations the soft-output LLR for the bit at position (1,1)?
Problem: Irregular LDPC code and decoder

An irregular LDPC code has 4 symbol nodes each of degree 2 and 3 check equations with degrees varying between 2 and 4.

A possible Tanner graph:

- Draw a possible Tanner graph.

Using the 4 received values: +1.0, +2.0, -1.0, +3.0 at an AWGN channel with 0 dB and perform the first steps in message passing with your Tanner graph.
Decoding with the simplified BCJR algorithm

Use a systematic feedforward memory one convolutional encoder

Perform the α and β recursion and determine the soft-output

The following 8 y values are received during the 4 sections:

+1.0, +2.0, +0.5, +1.5, -1.0, -1.5, -0.5, +2.0

Assume trellis starts at zero (+1) state followed by 3 sections and is terminated in the zero state (+1).

The following 8 y values are received during the 4 sections:

+1.0, +2.0, +0.5, +1.5, -1.0, -1.5, -0.5, +2.0

Use the metric with $\sum_T = \sum_T^c$
Problem: Repeat accumulate code

Describe the RA code as a parallel decodable system and explain its decoder functions.
Problem: BSC as test channel in EXIT charts

Prove that the ergodic mutual information chart formula gives the capacity of the BSC.
Given a source with alphabet A, B, C, D, E and the associated probabilities:

<table>
<thead>
<tr>
<th>Letter</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>1/2</td>
<td>1/4</td>
<td>1/8</td>
<td>1/16</td>
<td>1/16</td>
</tr>
</tbody>
</table>

Calculate entropy and average word length of the Huffman code.

Draw the Balakirsky trellis.

Is this trellis a good component for the iterative turbo scheme?

Balakirsky Trellis for Turbo Source Channel Decoding.