On the computation of A_∞-maps

A. Berciano, **M.J. Jiménez**, P. Real

Universidad de Sevilla

September 2007

Summary

Notion of A_∞-structures.

Summary

Notion of A_∞-structures.

Theoretical results to compute them.

Summary

Notion of A_∞-structures.

Theoretical results to compute them.

Improvements in the theoretical algorithms.

Index

Index

Rings

Let us suppose that Λ is a commutative ring with $1 \neq 0$.

Rings

Let us suppose that Λ is a commutative ring with $1 \neq 0$.

Modules

A *DG-module* is a graded module $M = \{M_n\}_{n \geq 0}$, endowed with a differential $d : M \to M$ (that is, a morphism of graded modules of degree -1 such that $d^2 = 0$).

Algebras

Definition

A differential graded algebra (A, μ_A, η), or simply DG-algebra, is a DG-module equipped with two morphisms $\mu_A : A \otimes A \to A$ and $\eta : \Lambda \to A$, such that μ_A is an *associative product*, i.e., $\mu_A(\mu_A \otimes 1) = \mu_A(1 \otimes \mu_A)$, and η is a bilateral unit $\eta : \Lambda \to A$, i.e.,

$$\begin{array}{ccccc}
\Lambda \otimes A & \xrightarrow{\cong} & A & \xrightarrow{\cong} & A \otimes \Lambda \\
\Big\downarrow{\scriptstyle \eta\otimes 1} & & \Big\Vert & & \Big\downarrow{\scriptstyle 1\otimes\eta} \\
A \otimes A & \xrightarrow[\mu_A]{} & A & \xleftarrow[\mu_A]{} & A \otimes A
\end{array}$$

A_∞-structures: Historical evolution

The notion of A_∞-algebras appears in the literature as a generalization of **"associative up to homotopy"**.

A_∞-structures: Historical evolution

The notion of A_∞-algebras appears in the literature as a generalization of **"associative up to homotopy"**.

Definition

Given M a differential graded module (DG-module), a morphism $\mu_2 : M \otimes M \rightarrow M$ of degree zero is said **associative up to homotopy** if

A_∞-structures: Historical evolution

The notion of A_∞-algebras appears in the literature as a generalization of **"associative up to homotopy"**.

Definition

Given M a differential graded module (DG-module), a morphism $\mu_2 : M \otimes M \rightarrow M$ of degree zero is said **associative up to homotopy** if

- it is compatible with the differential of M;

A_∞-structures: Historical evolution

The notion of A_∞-algebras appears in the literature as a generalization of **"associative up to homotopy"**.

Definition

Given M a differential graded module (DG-module), a morphism $\mu_2 : M \otimes M \to M$ of degree zero is said **associative up to homotopy** if

- it is compatible with the differential of M;
- there exists $\mu_3 : M^{\otimes 3} \to M$ of degree $+1$, such that

$$\mu_2(\mu_2 \otimes 1) - \mu_2(1 \otimes \mu_2).$$

A_∞-structures: Historical evolution

The notion of A_∞-algebras appears in the literature as a generalization of **"associative up to homotopy"**.

Definition

Given M a differential graded module (DG-module), a morphism $\mu_2 : M \otimes M \to M$ of degree zero is said **associative up to homotopy** if

- it is compatible with the differential of M;
- there exists $\mu_3 : M^{\otimes 3} \to M$ of degree $+1$, such that

$$\mu_2(\mu_2 \otimes 1) - \mu_2(1 \otimes \mu_2) \neq 0.$$

A_∞-structures: Historical evolution

The notion of A_∞-algebras appears in the literature as a generalization of **"associative up to homotopy"**.

Definition

Given M a differential graded module (DG-module), a morphism $\mu_2 : M \otimes M \to M$ of degree zero is said **associative up to homotopy** if

- it is compatible with the differential of M;
- there exists $\mu_3 : M^{\otimes 3} \to M$ of degree $+1$, such that

$$\mu_3 d + d\mu_3 = \mu_2(\mu_2 \otimes 1) - \mu_2(1 \otimes \mu_2) \neq 0.$$

A_∞-structures: Classical Example

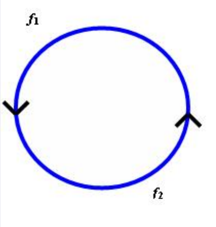
Let $(X, *)$ be a topological space with a base point $*$ and let ΩX denote the space of based loops in X: a point of ΩX is a continuous map $f : \mathbb{S}^1 \to X$ taking the base point of the circle to the base point $*$.

A_∞-structures: Classical Example

Let $(X, *)$ be a topological space with a base point $*$ and let ΩX denote the space of based loops in X: a point of ΩX is a continuous map $f : \mathbb{S}^1 \to X$ taking the base point of the circle to the base point $*$.

Let us take as multiplication the composition map

$$\mu_2 : \Omega X \times \Omega X \to \Omega X$$
$$(f_1, f_2) \longrightarrow f_1 * f_2$$

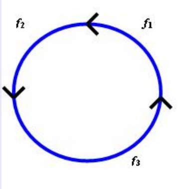


A_∞-structures: Classical Example

Non associative

μ_2 is not associative because of

$$(f_1 * f_2) * f_3$$

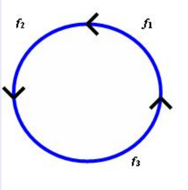


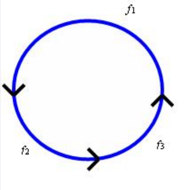
A_∞-structures: Classical Example

Non associative

μ_2 is not associative because of

$(f_1 * f_2) * f_3$ $\neq$ $f_1 * (f_2 * f_3)$



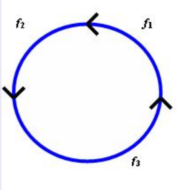


A_∞-structures: Classical Example

Non associative

μ_2 is associative up to homotopy

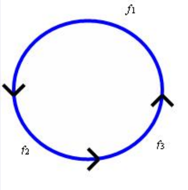
$(f_1 * f_2) * f_3$



Homotopy operator

$$\mu_3 : [0,1]\times(\Omega X)^3 \to \Omega X$$

$f_1 * (f_2 * f_3)$



A_∞-structures: Mathematical notion

Definition

An A_∞-algebra is a graded module A, with a family of graded maps $m_i : A^{\otimes i} \to A$, of degree $i-2$, such that for all $i \geq 1$:

$$\sum_{n=1}^{i} \sum_{k=0}^{i-n} (-1)^{n+k+nk} m_{i-n+1}(1^{\otimes k} \otimes m_n \otimes 1^{\otimes i-n-k}) = 0.$$

A_∞-structures: Mathematical notion

Definition

An A_∞-algebra is a graded module A, with a family of graded maps $m_i : A^{\otimes i} \to A$, of degree $i-2$, such that for all $i \geq 1$:

$$\sum_{n=1}^{i} \sum_{k=0}^{i-n} (-1)^{n+k+nk} m_{i-n+1}(1^{\otimes k} \otimes m_n \otimes 1^{\otimes i-n-k}) = 0.$$

- $m_1 m_1 = 0 \Rightarrow m_1$ is a differential.

A_∞-structures: Mathematical notion

Definition

An A_∞-algebra is a graded module A, with a family of graded maps $m_i : A^{\otimes i} \to A$, of degree $i-2$, such that for all $i \geq 1$:

$$\sum_{n=1}^{i}\sum_{k=0}^{i-n}(-1)^{n+k+nk} m_{i-n+1}(1^{\otimes k} \otimes m_n \otimes 1^{\otimes i-n-k}) = 0.$$

- $m_1 m_1 = 0 \Rightarrow m_1$ is a differential.
- $m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1) \Rightarrow m_2$ compatible with dif.

A_∞-structures: Mathematical notion

Definition

An A_∞-algebra is a graded module A, with a family of graded maps $m_i : A^{\otimes i} \to A$, of degree $i-2$, such that for all $i \geq 1$:

$$\sum_{n=1}^{i}\sum_{k=0}^{i-n}(-1)^{n+k+nk} m_{i-n+1}(1^{\otimes k} \otimes m_n \otimes 1^{\otimes i-n-k}) = 0.$$

- $m_1 m_1 = 0 \Rightarrow m_1$ is a differential.
- $m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1) \Rightarrow m_2$ compatible with dif.
- $m_3(m_1 \otimes 1^2 + 1^2 \otimes m_1 + 1 \otimes m_1 \otimes 1) + m_1 m_3 = m_2(m_2 \otimes 1 - 1 \otimes m_2)$.

A_∞-structures: Mathematical notion

Example

Every DG-algebra is, in particular, an A_∞-algebra with $m_1 = d$, $m_2 = \mu$ and $m_i = 0$ for all $i \geq 3$.

A_∞-structures: Mathematical notion

Example

Every DG-algebra is, in particular, an A_∞-algebra with $m_1 = d$, $m_2 = \mu$ and $m_i = 0$ for all $i \geq 3$.

Example

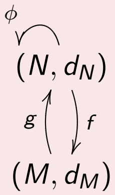
The chain complex of the loop space of X, $C_*(\Omega X)$ has an A_∞-algebra structure.

Index

Contractions

Definition

A *contraction* $c : \{N, M, f, g, \phi\}$ is a 5-tuple, such that

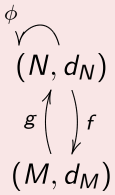


Contractions

Definition

A *contraction* $c : \{N, M, f, g, \phi\}$ is a 5-tuple, such that

- (N, d_N), (M, d_M) DG-modules.

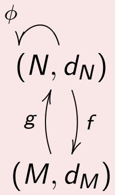


Contractions

Definition

A *contraction* $c : \{N, M, f, g, \phi\}$ is a 5-tuple, such that

- (N, d_N), (M, d_M) DG-modules.
- $f : N_* \longrightarrow M_*$ and $g : M_* \longrightarrow N_*$ morphisms of degree zero;

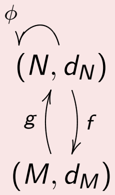


Contractions

Definition

A *contraction* $c : \{N, M, f, g, \phi\}$ is a 5-tuple, such that

- (N, d_N), (M, d_M) DG-modules.
- $f : N_* \longrightarrow M_*$ and $g : M_* \longrightarrow N_*$ morphisms of degree zero;
- $\phi : N_* \longrightarrow N_{*+1}$ is a homotopy operator;

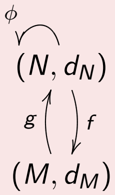


Contractions

Definition

A *contraction* $c : \{N, M, f, g, \phi\}$ is a 5-tuple, such that

- (N, d_N), (M, d_M) DG-modules.
- $f : N_* \longrightarrow M_*$ and $g : M_* \longrightarrow N_*$ morphisms of degree zero;
- $\phi : N_* \longrightarrow N_{*+1}$ is a homotopy operator;
- $fg = 1_M \, , \quad \phi d_N + d_N \phi + gf = 1_N;$

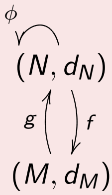


Contractions

Definition

A *contraction* $c : \{N, M, f, g, \phi\}$ is a 5-tuple, such that

- (N, d_N), (M, d_M) DG-modules.
- $f : N_* \longrightarrow M_*$ and $g : M_* \longrightarrow N_*$ morphisms of degree zero;
- $\phi : N_* \longrightarrow N_{*+1}$ is a homotopy operator;
- $fg = 1_M \,, \quad \phi d_N + d_N \phi + gf = 1_N$;
- $f\phi = 0 \,, \quad \phi g = 0 \,, \qquad \phi\phi = 0.$



Contractions: A_∞-structures

Theorem (Kad80,GLS91)

Let $(A,\, d_A,\, \mu)$ and $(M,\, d_M)$ be a connected DG–algebra and a DG–module, respectively and $c : \{A, M, f, g, \phi\}$ a contraction between them. Then the DG–module M is provided with an A_∞–algebra structure

Contractions: A_∞-structures

given by

$$m_n : M^{\otimes n} \to M$$

$$m_1 = -d_M$$

$$m_n = (-1)^{n+1} f\, \mu^{(1)}\, \phi^{[\otimes 2]}\, \mu^{(2)} \dots \phi^{[\otimes n-1]}\, \mu^{(n-1)}\, g^{\otimes n}\,, \quad n \geq 2 \quad (1)$$

with

$$\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1}\,,$$

$$\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g\, f)^{\otimes k-i-1}$$

Contractions: A_∞-structures

Computational Consequence

A contraction from a DG-algebra A to a DG-module M provides an algorithm to compute an A_∞-algebra structure on M.

Index

Main Results

Theorem

Any composition of the kind $\phi^{[\otimes s]}\mu^{(s)}$ ($s = 2, \ldots, n-1$) in the formula (1), which is given by

$$\left(\sum_{j=0}^{s-1} 1^{\otimes j} \otimes \phi \otimes (g\, f)^{\otimes s-j-1}\right) \circ \left(\sum_{i=0}^{s-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes s-i-1}\right),$$

can be reduced to

$$\phi^{[\otimes s]}\mu^{(s)} = \sum_{i=0}^{s-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi\mu_A \otimes 1^{\otimes s-i-1}.$$

Main Results

$$\phi^{[\otimes s]}\mu^{(s)} = \sum_{i=0}^{s-1}(-1)^{i+1}1^{\otimes i} \otimes \phi\mu_A \otimes 1^{\otimes s-i-1}\,. \qquad (2)$$

Main Results

$$\phi^{[\otimes s]}\mu^{(s)} = \sum_{i=0}^{s-1}(-1)^{i+1}1^{\otimes i} \otimes \phi\mu_A \otimes 1^{\otimes s-i-1}\,. \tag{2}$$

Moreover, given a composition of the kind

$$(\phi^{[\otimes s-1]}\mu^{(s-1)}) \circ (\phi^{[\otimes s]}\mu^{(s)}) \qquad s = 3, \ldots, n-2,$$

for every index i in the sum (2) of $\phi^{[\otimes s]}\mu^{(s)}$, the formula of $\phi^{[\otimes s-1]}\mu^{(s-1)}$ in such a composition can be reduced to

$$\sum_{j=i-1, j\geq 0}^{s-2} (-1)^{j+1}1^{\otimes j} \otimes \phi\mu_A \otimes 1^{\otimes s-j-2}\,. \tag{3}$$

Main Results

In other words, the whole composition $(\phi^{[\otimes 2]}\,\mu^{(2)}) \circ \cdots \circ (\phi^{[\otimes n-1]}\,\mu^{(n-1)})$ in the formula of m_n can be expressed by

$$\sum_{i_{n-1}=0}^{n-2} \left(\cdots \left(\sum_{i_2=i_3-1}^{1} (\phi\mu)^{(2,i_2)} \right) \cdots \right) (\phi\mu)^{(n-1,i_{n-1})},$$

where $(\phi\mu)^{(k,j)} = (-1)^{j+1} 1^{\otimes j} \otimes \phi\mu_A \otimes 1^{\otimes k-j-1}$ and each addend exists whenever the corresponding index $i_k \geq 0$.

Scheme of the proof

$$m_n = (-1)^{n+1} f\,\mu^{(1)}\;\underbrace{\phi^{[\otimes 2]}\,\mu^{(2)} \ldots \underbrace{\phi^{[\otimes n-2]}\,\mu^{(n-2)}\,\underbrace{\phi^{[\otimes n-1]}\,\mu^{(n-1)}\,g^{\otimes n}}_{k=1}}_{k=2}}_{k=n-2}$$

Scheme of the proof

$k = 1$

$$\phi^{[\otimes n-1]}\, \mu^{(n-1)}\, g^{\otimes n}$$

$$\left(\sum_{j=0}^{n-2} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes n-j-2} \right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \mu_A\, g^{\otimes 2} \otimes g^{\otimes n-i-2} \right)$$

Scheme of the proof

$\boxed{k=1}$ $\phi^{[\otimes n-1]}\,\mu^{(n-1)}\,g^{\otimes n}$

$$\left(\sum_{j=0}^{n-2} 1^{\otimes j}\otimes\phi\otimes(gf)^{\otimes n-j-2}\right)\circ\left(\sum_{i=0}^{n-2}\pm g^{\otimes i}\otimes\mu_A\,g^{\otimes 2}\otimes g^{\otimes n-i-2}\right)$$

$\phi\,g=0$
$f\,g=1$

Scheme of the proof

$\boxed{k=1}$ $\phi^{[\otimes n-1]}\,\mu^{(n-1)}\,g^{\otimes n}$

$$\left(\sum_{j=0}^{n-2} 1^{\otimes j} \otimes \phi \otimes (gf)^{\otimes n-j-2}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \mu_A\, g^{\otimes 2} \otimes g^{\otimes n-i-2}\right)$$

$\phi\, g = 0$
$f\, g = 1$ $\Rightarrow$

Scheme of the proof

$\boxed{k=1}$ $\phi^{[\otimes n-1]}\,\mu^{(n-1)}\,g^{\otimes n}$

$$\left(\sum_{j=0}^{n-2} 1^{\otimes j}\otimes\phi\otimes(gf)^{\otimes n-j-2}\right)\circ\left(\sum_{i=0}^{n-2}\pm g^{\otimes i}\otimes\mu_A\,g^{\otimes 2}\otimes g^{\otimes n-i-2}\right)$$

$\phi\,g=0$
$f\,g=1$ $\Rightarrow$

$$\boxed{\sum_{i=0}^{n-2}\pm 1^{\otimes i}\otimes\phi\,\mu_A\otimes 1^{\otimes n-i-2}}$$

Scheme of the proof

$\boxed{k=2}$

$$\left(\phi^{[\otimes n-2]}\,\mu^{(n-2)}\right)\circ\left(\sum_{i=0}^{n-2}\pm g^{\otimes i}\otimes\phi(sth.)\otimes g^{\otimes n-i-2}\right)$$

Scheme of the proof

$\boxed{k = 2}$

$$\left(\phi^{[\otimes n-2]}\,\mu^{(n-2)}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \phi(sth.) \otimes g^{\otimes n-i-2}\right)$$

$\phi\, g = 0$
$\phi\, \phi = 0$

Scheme of the proof

$\boxed{k=2}$

$$\left(\phi^{[\otimes n-2]}\,\mu^{(n-2)}\right)\circ\left(\sum_{i=0}^{n-2}\pm g^{\otimes i}\otimes\phi(sth.)\otimes g^{\otimes n-i-2}\right)$$

$\phi\, g = 0$
$\phi\,\phi = 0$ $\quad\Rightarrow$

Scheme of the proof

$\boxed{k=2}$

$$\left(\phi^{[\otimes n-2]}\,\mu^{(n-2)}\right)\circ\left(\sum_{i=0}^{n-2}\pm g^{\otimes i}\otimes\phi(sth.)\otimes g^{\otimes n-i-2}\right)$$

$$\begin{array}{l}\phi\, g=0\\ \phi\,\phi=0\end{array}\quad\Rightarrow\quad\sum_{j=0}^{n-3}\pm 1^{\otimes j}\otimes\phi\mu_A\otimes(gf)^{\otimes n-j-3}$$

Scheme of the proof

$k = 2$

$$\left(\phi^{[\otimes n-2]}\,\mu^{(n-2)}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \phi(sth.) \otimes g^{\otimes n-i-2}\right)$$

$$\phi\, g = 0$$
$$\phi\, \phi = 0$$

$$\Rightarrow \quad \sum_{j=0}^{n-3} \pm 1^{\otimes j} \otimes \phi\mu_A \otimes (gf)^{\otimes n-j-3}$$

$$f\, \phi = 0$$

Scheme of the proof

$\boxed{k=2}$

$$\left(\phi^{[\otimes n-2]}\,\mu^{(n-2)}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \phi(sth.) \otimes g^{\otimes n-i-2}\right)$$

$\phi\, g = 0$

$\phi\, \phi = 0 \qquad \Rightarrow \qquad \displaystyle\sum_{j=0}^{n-3} \pm 1^{\otimes j} \otimes \phi\mu_A \otimes (gf)^{\otimes n-j-3}$

$f\, \phi = 0 \qquad \Rightarrow$

Scheme of the proof

$\boxed{k = 2}$

$$\left(\phi^{[\otimes n-2]}\,\mu^{(n-2)}\right) \circ \left(\sum_{i=0}^{n-2} \pm g^{\otimes i} \otimes \phi(sth.) \otimes g^{\otimes n-i-2}\right)$$

$\phi\, g = 0$

$\phi\, \phi = 0$ $\Rightarrow$ $$\sum_{j=0}^{n-3} \pm 1^{\otimes j} \otimes \phi\mu_A \otimes (gf)^{\otimes n-j-3}$$

$f\, \phi = 0$ $\Rightarrow$ $$\boxed{\sum_{j=i-1}^{n-3} \pm 1^{\otimes j} \otimes \phi\mu_A \otimes 1^{\otimes n-j-3}}$$

Initial formulation

$$m_n = (-1)^{n+1} f\, \mu^{(1)}\, \phi^{[\otimes 2]}\, \mu^{(2)} \dots \phi^{[\otimes n-1]}\, \mu^{(n-1)}\, g^{\otimes n}\,, \quad n \geq 2\,.$$

Initial formulation

$$m_n = (-1)^{n+1} f\, \mu^{(1)}\, \phi^{[\otimes 2]}\, \mu^{(2)} \dots \phi^{[\otimes n-1]}\, \mu^{(n-1)}\, g^{\otimes n}\,, \quad n \geq 2\,.$$

- $\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g\, f)^{\otimes k-i-1} \longrightarrow k$ addends.

Initial formulation

$$m_n = (-1)^{n+1} f\, \mu^{(1)}\, \phi^{[\otimes 2]}\, \mu^{(2)} \dots \phi^{[\otimes n-1]}\, \mu^{(n-1)}\, g^{\otimes n}\,, \quad n \geq 2\,.$$

- $\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g\, f)^{\otimes k-i-1} \longrightarrow k$ addends.
- $\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1} \longrightarrow k$ addends.

Initial formulation

$$m_n = (-1)^{n+1} f\, \mu^{(1)}\, \phi^{[\otimes 2]}\, \mu^{(2)} \dots \phi^{[\otimes n-1]}\, \mu^{(n-1)}\, g^{\otimes n}\,, \quad n \geq 2\,.$$

- $\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g\, f)^{\otimes k-i-1} \longrightarrow k$ addends.
- $\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1} \longrightarrow k$ addends.
- $\phi^{[\otimes k]}\, \mu^{(k)}$ contributes with k^2 addends.

Initial formulation

$$m_n = (-1)^{n+1} f\, \mu^{(1)}\, \phi^{[\otimes 2]}\, \mu^{(2)} \dots \phi^{[\otimes n-1]}\, \mu^{(n-1)}\, g^{\otimes n}\,, \quad n \geq 2\,.$$

- $\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g\,f)^{\otimes k-i-1} \longrightarrow k$ addends.
- $\mu^{(k)} = \displaystyle\sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1} \longrightarrow k$ addends.
- $\phi^{[\otimes k]}\, \mu^{(k)}$ contributes with k^2 addends.
- m_n is $O((n-1)!^2)$ in space.

Initial formulation

$$m_n = (-1)^{n+1} f\, \mu^{(1)}\, \phi^{[\otimes 2]}\, \mu^{(2)} \dots \phi^{[\otimes n-1]}\, \mu^{(n-1)}\, g^{\otimes n}\,, \quad n \geq 2\,.$$

- $\phi^{[\otimes k]} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes \phi \otimes (g\, f)^{\otimes k-i-1} \longrightarrow k$ addends.
- $\mu^{(k)} = \displaystyle\sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1} \longrightarrow k$ addends.
- $\phi^{[\otimes k]}\, \mu^{(k)}$ contributes with k^2 addends.
- m_n is $O((n-1)!^2)$ in space.

The number of basic operations can be expressed by $n + n\,(n-1)!^2 + \frac{(n+3)(n-2)}{4}\,(n-1)!^2$, so m_n is $O((n)!^2)$ in time.

First reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{i=0}^{k-1}(-1)^{i+1}1^{\otimes i} \otimes \phi\mu_A \otimes 1^{\otimes k-i-1}\,.$$

First reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{i=0}^{k-1}(-1)^{i+1}1^{\otimes i} \otimes \phi\mu_A \otimes 1^{\otimes k-i-1}\,.$$

$\phi^{[\otimes k]}\,\mu^{(k)}$: k^2 addends $\longrightarrow$ k addends.
So, now m_n is $O((n-1)!)$ in space.

First reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \phi\mu_A \otimes 1^{\otimes k-i-1}\,.$$

$\phi^{[\otimes k]}\,\mu^{(k)}$: k^2 addends $\longrightarrow$ k addends.
So, now m_n is $O((n-1)!)$ in space.

Now, the number of operations is exactly

$$n + (n-1)!\,(2n-2)\,,$$

so, m_n is $O((n)!)$ in time.

Second reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{j=i-1,\, j\geq 0}^{k-2} (-1)^{j+1} 1^{\otimes j} \otimes \phi\mu_A \otimes 1^{\otimes k-j-2}\,.$$

Second reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{j=i-1,\, j\geq 0}^{k-2} (-1)^{j+1} 1^{\otimes j} \otimes \phi\mu_A \otimes 1^{\otimes k-j-2}\,.$$

The number of addends becomes $(n-1)! - S_n$,

$$\frac{(n-1)!}{2} < (n-1)! - S_n < (n-1)!\,,$$

But $(n-1)! - S_n$ is much "closer" to $\frac{(n-1)!}{2}$ than to $(n-1)!$:

Second reduction

$$\phi^{[\otimes k]}\mu^{(k)} = \sum_{j=i-1,\, j\geq 0}^{k-2} (-1)^{j+1} 1^{\otimes j} \otimes \phi\mu_A \otimes 1^{\otimes k-j-2}\,.$$

The number of addends becomes $(n-1)! - S_n$,

$$\frac{(n-1)!}{2} < (n-1)! - S_n < (n-1)!\,,$$

But $(n-1)! - S_n$ is much "closer" to $\frac{(n-1)!}{2}$ than to $(n-1)!$:

n	5	10	50	100
$((n-1)! - S_n)/(n-1)!$	$0,70833$	$0,5637$	$0,51042$	$0,5051$

Comparative table

Summing up,

	original formula		**new formula**	
	time	space	time	space
m_n	$O(n!^2)$	$O((n-1)!^2)$	$O(n!)$	$O((n-1)!)$